

INVARIANT SUBALGEBRAS OF AFFINE VERTEX ALGEBRAS

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Dedicated to my father Michael A. Linshaw, M. D., on the occasion of his 70th birthday

ABSTRACT. Given a finite-dimensional complex Lie algebra \mathfrak{g} equipped with a nondegenerate, symmetric, invariant bilinear form B , let $V_k(\mathfrak{g}, B)$ denote the universal affine vertex algebra associated to \mathfrak{g} and B at level k . For any reductive group G of automorphisms of $V_k(\mathfrak{g}, B)$, we show that the invariant subalgebra $V_k(\mathfrak{g}, B)^G$ is strongly finitely generated for generic values of k . This implies the existence of a new family of deformable \mathcal{W} -algebras $\mathcal{W}(\mathfrak{g}, B, G)_k$ which exist for all but finitely many values of the parameter k .

1. INTRODUCTION

We call a vertex algebra \mathcal{V} *strongly finitely generated* if there exists a finite set of generators such that the collection of iterated Wick products of the generators and their derivatives spans \mathcal{V} . Many known vertex algebras have this property, including affine, free field and lattice vertex algebras, as well as the \mathcal{W} -algebras $\mathcal{W}(\mathfrak{g}, f)_k$ associated via quantum Drinfeld-Sokolov reduction to a simple, finite-dimensional Lie algebra \mathfrak{g} and a nilpotent element $f \in \mathfrak{g}$. Strong finite generation has many important consequences, and in particular implies that both Zhu's associative algebra $A(\mathcal{V})$, and Zhu's commutative algebra $\mathcal{V}/C_2(\mathcal{V})$, are finitely generated.

In recent work, we have investigated the strong finite generation of invariant vertex algebras \mathcal{V}^G , where \mathcal{V} is simple and G is a reductive group of automorphisms of \mathcal{V} . This is a vertex algebra analogue of Hilbert's theorem on the finite generation of classical invariant rings. It is a subtle and essentially "quantum" phenomenon that is generally destroyed by passing to the classical limit before taking invariants. Often, \mathcal{V} admits a G -invariant filtration for which $gr(\mathcal{V})$ is a commutative algebra with a derivation (i.e., an abelian vertex algebra), and the classical limit $gr(\mathcal{V}^G)$ is isomorphic to $(gr(\mathcal{V}))^G$ as a commutative algebra. Unlike \mathcal{V}^G , $gr(\mathcal{V}^G)$ is generally not finitely generated as a vertex algebra, and a presentation will require both infinitely many generators and infinitely many relations.

Isolated examples of this phenomenon have been known for some years (see for example [BFH][EFH][DN][FKRW][KWY]), although the first general results of this kind were obtained by the author in [LII], in the case where \mathcal{V} is the $\beta\gamma$ -system $\mathcal{S}(V)$ associated to the vector space $V = \mathbb{C}^n$. The full automorphism group of $\mathcal{S}(V)$ preserving a natural conformal structure is GL_n . By a theorem of Kac-Radul [KR], $\mathcal{S}(V)^{GL_n}$ is isomorphic to the vertex algebra $\mathcal{W}_{1+\infty}$ with central charge $-n$. In [LI] we showed that $\mathcal{W}_{1+\infty, -n}$ has a minimal strong generating set consisting of $n^2 + 2n$ elements, and in particular is a \mathcal{W} -algebra of type $\mathcal{W}(1, 2, \dots, n^2 + 2n)$. For an arbitrary reductive group $G \subset GL_n$, $\mathcal{S}(V)^G$ decomposes as a direct sum of irreducible, highest-weight $\mathcal{W}_{1+\infty, -n}$ -modules. The strong finite

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generation of $\mathcal{W}_{1+\infty, -n}$ implies a certain finiteness property of the modules appearing in $\mathcal{S}(V)^G$. This property, together with a classical theorem of Weyl, yielded the strong finite generation of $\mathcal{S}(V)^G$. Using the same approach, we also proved in [LII] that invariant subalgebras of bc -systems and $bc\beta\gamma$ -systems are strongly finitely generated.

In [LIII] we initiated a similar study of the invariant subalgebras of the rank n Heisenberg vertex algebra $\mathcal{H}(n)$. The full automorphism group of $\mathcal{H}(n)$ preserving a natural conformal structure is the orthogonal group $O(n)$. Motivated by classical invariant theory, we conjectured that $\mathcal{H}(n)^{O(n)}$ is a \mathcal{W} -algebra of type $\mathcal{W}(2, 4, \dots, n^2 + 3n)$. For $n = 1$, this was already known to Dong-Nagatomo [DN], and we proved it for $n = 2$ and $n = 3$. We also showed that this conjecture implies the strong finite generation of $\mathcal{H}(n)^G$ for an arbitrary reductive group G .

In this paper, we study invariant subalgebras of the universal affine vertex algebra $V_k(\mathfrak{g}, B)$ for a finite-dimensional Lie algebra \mathfrak{g} equipped with a nondegenerate, symmetric, invariant bilinear form B . Recall that $V_k(\mathfrak{g}, B)$ has generators X^ξ , which are linear in $\xi \in \mathfrak{g}$, and satisfy the OPE relations

$$X^\xi(z)X^\eta(w) \sim kB(\xi, \eta)(z - w)^{-2} + X^{[\xi, \eta]}(w)(z - w)^{-1}.$$

Let G be a reductive group of automorphisms of $V_k(\mathfrak{g}, B)$ for all $k \in \mathbb{C}$. Our main result is

Theorem 1.1. *For any \mathfrak{g} , B , and G , $V_k(\mathfrak{g}, B)^G$ is strongly finitely generated for generic values of k , i.e., for $k \in \mathbb{C} \setminus K$ where K is at most countable.*

A *deformable \mathcal{W} -algebra* is a family of vertex algebras \mathcal{W}_k depending on a parameter k , equipped with strong generating sets $A_k = \{a_1^k, \dots, a_r^k\}$, whose structure constants are continuous functions of k with isolated singularities. It is customary in the physics literature to assume that one of the generators is a Virasoro element (with central charge depending continuously on k) and the remaining generators are primary, but in our definition these conditions have been relaxed. The structure constants here are the coefficients of each normally ordered monomial in the elements of A_k and their derivatives, which appear in the OPE of $a_i^k(z)a_j^k(w)$, for $i, j = 1, \dots, r$. An easy consequence of Theorem 1.1 is

Corollary 1.2. *For any \mathfrak{g} , B , and G as above, there is a deformable \mathcal{W} -algebra $\mathcal{W}(\mathfrak{g}, B, G)_k$ which exists for all but finitely many values of k . For generic values of k , $\mathcal{W}(\mathfrak{g}, B, G)_k = V_k(\mathfrak{g}, B)^G$.*

Perhaps the best-studied examples of deformable \mathcal{W} -algebras are the algebras $\mathcal{W}(\mathfrak{g}, f)_k$ associated to a simple, finite-dimensional Lie algebra \mathfrak{g} , and a nilpotent element $f \in \mathfrak{g}$ (see [KRW] and references therein). These algebras have the property that they are *freely generated*; there exists a strong finite generating set such that there are no nontrivial normally ordered polynomial relations among the generators and their derivatives (see [DSK]). By contrast, $\mathcal{W}(\mathfrak{g}, B, G)_k$ is generally *not* freely generated for the simple reason that $gr(V_k(\mathfrak{g}, B))^G \cong (Sym \bigoplus_{j \geq 0} V_j)^G$, which is almost never a polynomial algebra. These vertex algebras form a new and rich class of deformable \mathcal{W} -algebras, and it is important to study their representation theory as well as their structure. For generic values of k , $\mathcal{W}(\mathfrak{g}, B, G)_k$ will be simple, but for special values of k these algebras will possess nontrivial ideals. It is likely that for positive integer values of k , there exist new rational vertex algebras arising as quotients of vertex algebras of this kind.

The proof of Theorem 1.1 is divided into three steps. The first step is to prove it in the special case where \mathfrak{g} is *abelian*, so that $V_k(\mathfrak{g}, B) \cong \mathcal{H}(n)$ for $n = \dim(\mathfrak{g})$, and G is the *full*

automorphism group $O(n)$ of $\mathcal{H}(n)$. The second step (which is a minor modification of Theorem 6.9 of [LIII]) is to show that the strong finite generation of $\mathcal{H}(n)^{O(n)}$ implies the strong finite generation of $\mathcal{H}(n)^G$ for an arbitrary reductive group G . The third step is to reduce the general case to the case where \mathfrak{g} is abelian. More precisely, if G acts on $V_k(\mathfrak{g}, B)$ for all k , G acts on the weight-one subspace $V_k(\mathfrak{g}, B)[1] \cong \mathfrak{g}$, and G preserves both the bracket and the bilinear form on \mathfrak{g} . Therefore $G \subset O(n)$ and G also acts on $\mathcal{H}(n)$. Both $V_k(\mathfrak{g}, B)^G$ and $\mathcal{H}(n)^G$ admit G -invariant filtrations, and we have linear isomorphisms

$$(1.1) \quad \mathcal{H}(n)^G \cong \text{gr}(\mathcal{H}(n)^G) \cong \text{gr}(\mathcal{H}(n))^G \cong \text{gr}(V_k(\mathfrak{g}, B)^G) \cong \text{gr}(V_k(\mathfrak{g}, B))^G \cong V_k(\mathfrak{g}, B)^G,$$

and isomorphisms of graded commutative rings

$$(1.2) \quad \text{gr}(V_k(\mathfrak{g}, B))^G \cong (\text{Sym} \bigoplus_{j \geq 0} V_j)^G \cong \text{gr}(\mathcal{H}(n))^G.$$

Here $V_j \cong \mathbb{C}^n \cong \mathfrak{g}$ as G -modules. Therefore both $V_k(\mathfrak{g}, B)^G$ and $\mathcal{H}(n)^G$ can be viewed as deformations of the *same* classical invariant ring $R = (\text{Sym} \bigoplus_{j \geq 0} V_j)^G$. By a theorem of Weyl, there is a natural (infinite) generating set S for $(\text{Sym} \bigoplus_{j \geq 0} V_j)^G$ consisting of a finite set of generators for $(\text{Sym} \bigoplus_{j=0}^{n-1} V_j)^G$ together with their polarizations. Under (1.1) and (1.2), S corresponds to (infinite) strong generating sets T and U for $\mathcal{H}(n)^G$ and $V_k(\mathfrak{g}, B)^G$, respectively.

Since $\mathcal{H}(n)^G$ is strongly finitely generated, we may assume without loss of generality that there is a finite subset $T' = \{p_1, \dots, p_s\} \subset T$, which strongly generates $\mathcal{H}(n)^G$. In other words, $\mathcal{H}(n)^G = \langle T' \rangle$, where $\langle T' \rangle$ denotes the space of normally ordered polynomials in p_1, \dots, p_s and their derivatives. This implies that any $q \in T$ admits a “decoupling relation”

$$(1.3) \quad q = P(p_1, \dots, p_s),$$

where P lies in $\langle T' \rangle$. The relations in $(\text{Sym} \bigoplus_{j \geq 0} V_j)^G$ among the generators in S are given by the *second fundamental theorem of invariant theory* for (G, V) . We may view the decoupling relation (1.3) as a deformation of such a classical relation.

Let $U' \subset U \subset V_k(\mathfrak{g}, B)^G$ be the set corresponding to T' under (1.1), which we denote by $\{\tilde{p}_1, \dots, \tilde{p}_s\}$, and let $\langle U' \rangle$ denote the space of normally ordered polynomials in $\tilde{p}_1, \dots, \tilde{p}_s$ and their derivatives. Given $q \in T$, let $\tilde{q} \in U$ be the element corresponding to q under (1.1). We show that for generic values of k , the decoupling relations (1.3) can be used to construct analogous (but more complicated) decoupling relations

$$(1.4) \quad \tilde{q} = Q(\tilde{p}_1, \dots, \tilde{p}_s),$$

in $V_k(\mathfrak{g}, B)^G$ for all $\tilde{q} \in U$, where Q lies in $\langle U' \rangle$. This implies that U' is a strong generating set for $V_k(\mathfrak{g}, B)^G$ for generic values of k .

Finally, let $\mathcal{W}(\mathfrak{g}, B, G)_k$ be the vertex subalgebra of $V_k(\mathfrak{g}, B)^G$ generated by U' . Clearly $\langle U' \rangle \subset \mathcal{W}(\mathfrak{g}, B, G)_k$ for all k , and the equality $\langle U' \rangle = \mathcal{W}(\mathfrak{g}, B, G)_k$ holds precisely when $\mathcal{W}(\mathfrak{g}, B, G)_k$ is strongly generated by U' . We will see that $\mathcal{W}(\mathfrak{g}, B, G)_k$ is a deformable \mathcal{W} -algebra whose structure constants are rational functions of k , so $\langle U' \rangle = \mathcal{W}(\mathfrak{g}, B, G)_k$ for all but finitely many values of k .

The idea of studying $V_k(\mathfrak{g}, B)^G$ via the simpler object $\mathcal{H}(n)^G$ is analogous to studying classical invariant rings of the form $U(\mathfrak{g})^G$ via the Poincaré-Birkhoff-Witt abelianization $\text{Sym}(\mathfrak{g})^G \cong \text{gr}(U(\mathfrak{g})^G) \cong \text{gr}(U(\mathfrak{g}))^G$. Here $U(\mathfrak{g})$ denotes the universal enveloping algebra

of \mathfrak{g} , and G is a group of automorphisms of $U(\mathfrak{g})$. A generating set for $U(\mathfrak{g})^G$ can be obtained from a generating set for $Sym(\mathfrak{g})^G$, and the leading term of a relation among the generators of $U(\mathfrak{g})^G$ corresponds to a relation among the generators of $Sym(\mathfrak{g})^G$. However, more subtle information about the structure and representation theory of $U(\mathfrak{g})^G$ cannot be reconstructed in this way. Likewise, we can view $\mathcal{H}(n)$ as a kind of “partial abelianization” of $V_k(\mathfrak{g}, B)$. By passing to the full abelianization $gr(V_k(\mathfrak{g}, B)) \cong Sym \bigoplus_{j \geq 0} V_j$ of $V_k(\mathfrak{g}, B)$, we destroy too much structure, and $gr(V_k(\mathfrak{g}, B))^G$ generally fails to be finitely generated. By contrast, $\mathcal{H}(n)$ retains enough structure (notably, it is still a simple vertex algebra) so that strong generating sets for $\mathcal{H}(n)^G$ and $V_k(\mathfrak{g}, B)^G$ are closely related.

2. VERTEX ALGEBRAS

In this section, we define vertex algebras, which have been discussed from various different points of view in the literature (see for example [B][FLM][K][FBZ]). We will follow the formalism developed in [LZI] and partly in [Li]. Let $V = V_0 \oplus V_1$ be a super vector space over \mathbb{C} , and let z, w be formal variables. By $QO(V)$, we mean the space of all linear maps

$$V \rightarrow V((z)) := \left\{ \sum_{n \in \mathbb{Z}} v(n) z^{-n-1} \mid v(n) \in V, v(n) = 0 \text{ for } n \gg 0 \right\}.$$

Each element $a \in QO(V)$ can be uniquely represented as a power series

$$a = a(z) := \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in (End V)[[z, z^{-1}]].$$

We refer to $a(n)$ as the n th Fourier mode of $a(z)$. Each $a \in QO(V)$ is assumed to be of the shape $a = a_0 + a_1$ where $a_i : V_j \rightarrow V_{i+j}((z))$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$, and we write $|a_i| = i$.

On $QO(V)$ there is a set of nonassociative bilinear operations \circ_n , indexed by $n \in \mathbb{Z}$, which we call the n th circle products. For homogeneous $a, b \in QO(V)$, they are defined by

$$a(w) \circ_n b(w) = Res_z a(z) b(w) \iota_{|z| > |w|} (z - w)^n - (-1)^{|a||b|} Res_z b(w) a(z) \iota_{|w| > |z|} (z - w)^n.$$

Here $\iota_{|z| > |w|} f(z, w) \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]]$ denotes the power series expansion of a rational function f in the region $|z| > |w|$. We usually omit the symbol $\iota_{|z| > |w|}$ and just write $(z - w)^{-1}$ to mean the expansion in the region $|z| > |w|$, and write $-(w - z)^{-1}$ to mean the expansion in $|w| > |z|$. It is easy to check that $a(w) \circ_n b(w)$ above is a well-defined element of $QO(V)$.

The non-negative circle products are connected through the *operator product expansion* (OPE) formula. For $a, b \in QO(V)$, we have

$$(2.1) \quad a(z)b(w) = \sum_{n \geq 0} a(w) \circ_n b(w) (z - w)^{-n-1} + : a(z)b(w) : ,$$

which is often written as $a(z)b(w) \sim \sum_{n \geq 0} a(w) \circ_n b(w) (z - w)^{-n-1}$, where \sim means equal modulo the term

$$: a(z)b(w) : = a(z)_- b(w) + (-1)^{|a||b|} b(w) a(z)_+.$$

Here $a(z)_- = \sum_{n < 0} a(n) z^{-n-1}$ and $a(z)_+ = \sum_{n \geq 0} a(n) z^{-n-1}$. Note that $: a(w)b(w) :$ is a well-defined element of $QO(V)$. It is called the *Wick product* of a and b , and it coincides

with $a \circ_{-1} b$. The other negative circle products are related to this by

$$n! a(z) \circ_{-n-1} b(z) = : (\partial^n a(z)) b(z) :,$$

where ∂ denotes the formal differentiation operator $\frac{d}{dz}$. For $a_1(z), \dots, a_k(z) \in QO(V)$, the k -fold iterated Wick product is defined to be

$$(2.2) \quad : a_1(z) a_2(z) \cdots a_k(z) : = : a_1(z) b(z) :,$$

where $b(z) = : a_2(z) \cdots a_k(z) : .$ We often omit the formal variable z when no confusion will arise.

The set $QO(V)$ is a nonassociative algebra with the operations \circ_n and a unit 1. We have $1 \circ_n a = \delta_{n,-1} a$ for all n , and $a \circ_n 1 = \delta_{n,-1} a$ for $n \geq -1$. A linear subspace $\mathcal{A} \subset QO(V)$ containing 1 which is closed under the circle products will be called a quantum operator algebra (QOA). In particular \mathcal{A} is closed under ∂ since $\partial a = a \circ_{-2} 1$. Many formal algebraic notions are immediately clear: a homomorphism is just a linear map that sends 1 to 1 and preserves all circle products; a module over \mathcal{A} is a vector space M equipped with a homomorphism $\mathcal{A} \rightarrow QO(M)$, etc. A subset $S = \{a_i \mid i \in I\}$ of \mathcal{A} is said to *generate* \mathcal{A} if any element $a \in \mathcal{A}$ can be written as a linear combination of nonassociative words in the letters a_i, \circ_n , for $i \in I$ and $n \in \mathbb{Z}$. We say that S *strongly generates* \mathcal{A} if any $a \in \mathcal{A}$ can be written as a linear combination of words in the letters a_i, \circ_n for $n < 0$. Equivalently, \mathcal{A} is spanned by the collection $\{ : \partial^{k_1} a_{i_1}(z) \cdots \partial^{k_m} a_{i_m}(z) : \mid i_1, \dots, i_m \in I, k_1, \dots, k_m \geq 0 \}$.

We say that $a, b \in QO(V)$ *quantum commute* if $(z - w)^N [a(z), b(w)] = 0$ for some $N \geq 0$. Here $[,]$ denotes the super bracket. This condition implies that $a \circ_n b = 0$ for $n \geq N$, so (2.1) becomes a finite sum. If N can be chosen to be 0, we say that a, b commute. A commutative quantum operator algebra (CQOA) is a QOA whose elements pairwise quantum commute. Finally, the notion of a CQOA is equivalent to the notion of a vertex algebra. Every CQOA \mathcal{A} is itself a faithful \mathcal{A} -module, called the *left regular module*. Define

$$\rho : \mathcal{A} \rightarrow QO(\mathcal{A}), \quad a \mapsto \hat{a}, \quad \hat{a}(\zeta)b = \sum_{n \in \mathbb{Z}} (a \circ_n b) \zeta^{-n-1}.$$

Then ρ is an injective QOA homomorphism, and the quadruple of structures $(\mathcal{A}, \rho, 1, \partial)$ is a vertex algebra in the sense of [FLM]. Conversely, if $(V, Y, 1, D)$ is a vertex algebra, the collection $Y(V) \subset QO(V)$ is a CQOA. We will refer to a CQOA simply as a vertex algebra throughout the rest of this paper.

The main examples we consider are the *universal affine vertex algebras* and their invariant subalgebras. Let \mathfrak{g} be a finite-dimensional Lie algebra over \mathbb{C} , equipped with a nondegenerate, symmetric, invariant bilinear form B . The loop algebra $\mathfrak{g}[t, t^{-1}] = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ has a one-dimensional central extension $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}\kappa$ determined by B , with bracket

$$[\xi t^n, \eta t^m] = [\xi, \eta] t^{n+m} + nB(\xi, \eta) \delta_{n+m, 0} \kappa,$$

and \mathbb{Z} -gradation $\deg(\xi t^n) = n$, $\deg(\kappa) = 0$. Let $\hat{\mathfrak{g}}_{\geq 0} = \bigoplus_{n \geq 0} \hat{\mathfrak{g}}_n$ where $\hat{\mathfrak{g}}_n$ denotes the subspace of degree n . For $k \in \mathbb{C}$, let C_k be the one-dimensional $\hat{\mathfrak{g}}_{\geq 0}$ -module on which ξt^n acts trivially for $n \geq 0$, and κ acts by the k times the identity. Define $V_k = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{\geq 0})} C_k$, and let $X^\xi(n) \in \text{End}(V_k)$ be the linear operator representing ξt^n on V_k . Define $X^\xi(z) = \sum_{n \in \mathbb{Z}} X^\xi(n) z^{-n-1}$, which is easily seen to lie in $QO(V_k)$ and satisfy the OPE relation

$$X^\xi(z) X^\eta(w) \sim kB(\xi, \eta)(z - w)^{-2} + X^{[\xi, \eta]}(w)(z - w)^{-1}.$$

The vertex algebra $V_k(\mathfrak{g}, B)$ generated by $\{X^\xi \mid \xi \in \mathfrak{g}\}$ is known as the universal affine vertex algebra associated to \mathfrak{g} and B at level k .

Two special cases will be important to us. First, suppose that \mathfrak{g} is simple, so that B is a scalar multiple of the normalized Killing form $\langle, \rangle = \frac{1}{2h^\vee} \langle, \rangle_K$, for some scalar k . In this case, it is customary to denote $V_k(\mathfrak{g}, B)$ by $V_k(\mathfrak{g})$. Let $n = \dim(\mathfrak{g})$, and fix an orthonormal basis $\{\xi_1, \dots, \xi_n\}$ for \mathfrak{g} relative to \langle, \rangle . For $k \neq -h^\vee$ where h^\vee is the dual Coxeter number, $V_k(\mathfrak{g})$ is a conformal vertex algebra with Virasoro element

$$L(z) = \frac{1}{2(k + h^\vee)} \sum_{i=1}^n : X^{\xi_i}(z) X^{\xi_i}(z) :,$$

of central charge $\frac{k \cdot \dim(\mathfrak{g})}{k + h^\vee}$, such that each X^{ξ_i} is primary of weight one. This Virasoro element is known as the *Sugawara conformal vector*, and it lies in $V_k(\mathfrak{g})^G$ for any group G of automorphisms of $V_k(\mathfrak{g})$. At the critical level $k = -h^\vee$, $L(z)$ does not exist, but $V_k(\mathfrak{g})$ still possesses a *quasi-conformal structure*; there is an action of the Lie subalgebra $\{L_n \mid n \geq -1\}$ of the Virasoro algebra, such that L_{-1} acts by translation, L_0 acts diagonalizably, and each X^{ξ_i} has weight one. In fact, this quasiconformal structure exists on $V_k(\mathfrak{g}, B)$ for any \mathfrak{g} and B . We will always assume that $V_k(\mathfrak{g}, B)$ is equipped with this conformal weight grading; note that the weight-one subspace $V_k(\mathfrak{g}, B)[1]$ is then linearly isomorphic to \mathfrak{g} . Any group of automorphisms of $V_k(\mathfrak{g}, B)$ preserving this grading must act on \mathfrak{g} and preserve both the bracket and the bilinear form.

Next, suppose that \mathfrak{g} is abelian. Since B is nondegenerate, $V_k(\mathfrak{g}, B)$ is just the rank n Heisenberg vertex algebra $\mathcal{H}(n)$. If we choose an orthonormal basis $\{\xi_1, \dots, \xi_n\}$ for \mathfrak{g} , $\mathcal{H}(n)$ is generated by $\{\alpha^i = X^{\xi_i} \mid i = 1, \dots, n\}$, satisfying the OPE relations

$$\alpha^i(z) \alpha^j(w) \sim \delta_{i,j} (z - w)^{-2}.$$

There is a conformal structure of central charge n on $\mathcal{H}(n)$, with Virasoro element

$$L(z) = \frac{1}{2} \sum_{i=1}^n : \alpha^i(z) \alpha^i(z) :,$$

under which each α^i is primary of weight one.

3. CATEGORY \mathcal{R}

Let \mathcal{R} be the category of vertex algebras \mathcal{A} equipped with a $\mathbb{Z}_{\geq 0}$ -filtration

$$(3.1) \quad \mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} \subset \dots, \quad \mathcal{A} = \bigcup_{k \geq 0} \mathcal{A}_{(k)}$$

such that $\mathcal{A}_{(0)} = \mathbb{C}$, and for all $a \in \mathcal{A}_{(k)}$, $b \in \mathcal{A}_{(l)}$, we have

$$(3.2) \quad a \circ_n b \in \mathcal{A}_{(k+l)}, \quad \text{for } n < 0,$$

$$(3.3) \quad a \circ_n b \in \mathcal{A}_{(k+l-1)}, \quad \text{for } n \geq 0.$$

Elements $a(z) \in \mathcal{A}_{(d)} \setminus \mathcal{A}_{(d-1)}$ are said to have degree d .

Filtrations on vertex algebras satisfying (3.2)-(3.3) were introduced in [Li], and are known as *good increasing filtrations*. Setting $\mathcal{A}_{(-1)} = \{0\}$, the associated graded object $gr(\mathcal{A}) = \bigoplus_{k \geq 0} \mathcal{A}_{(k)} / \mathcal{A}_{(k-1)}$ is a $\mathbb{Z}_{\geq 0}$ -graded associative, (super)commutative algebra with

a unit 1 under a product induced by the Wick product on \mathcal{A} . In general, there is no natural linear map $\mathcal{A} \rightarrow gr(\mathcal{A})$, but for each $r \geq 1$ we have the projection

$$(3.4) \quad \phi_r : \mathcal{A}_{(r)} \rightarrow \mathcal{A}_{(r)} / \mathcal{A}_{(r-1)} \subset gr(\mathcal{A}).$$

Moreover, $gr(\mathcal{A})$ has a derivation ∂ of degree zero (induced by the operator $\partial = \frac{d}{dz}$ on \mathcal{A}), and for each $a \in \mathcal{A}_{(d)}$ and $n \geq 0$, the operator $a \circ_n$ on \mathcal{A} induces a derivation of degree $d - k$ on $gr(\mathcal{A})$, which we denote by $a(n)$. Here

$$k = \sup\{j \geq 1 \mid \mathcal{A}_{(r)} \circ_n \mathcal{A}_{(s)} \subset \mathcal{A}_{(r+s-j)} \ \forall r, s, n \geq 0\},$$

as in [LL]. Finally, these derivations give $gr(\mathcal{A})$ the structure of a vertex Poisson algebra.

The assignment $\mathcal{A} \mapsto gr(\mathcal{A})$ is a functor from \mathcal{R} to the category of $\mathbb{Z}_{\geq 0}$ -graded (super)commutative rings with a differential ∂ of degree 0, which we will call ∂ -rings. A ∂ -ring is the same thing as an *abelian* vertex algebra, that is, a vertex algebra \mathcal{V} in which $[a(z), b(w)] = 0$ for all $a, b \in \mathcal{V}$. A ∂ -ring A is said to be generated by a subset $\{a_i \mid i \in I\}$ if $\{\partial^k a_i \mid i \in I, k \geq 0\}$ generates A as a graded ring. The key feature of \mathcal{R} is the following reconstruction property [LL]:

Lemma 3.1. *Let \mathcal{A} be a vertex algebra in \mathcal{R} and let $\{a_i \mid i \in I\}$ be a set of generators for $gr(\mathcal{A})$ as a ∂ -ring, where a_i is homogeneous of degree d_i . If $a_i(z) \in \mathcal{A}_{(d_i)}$ are vertex operators such that $\phi_{d_i}(a_i(z)) = a_i$, then \mathcal{A} is strongly generated as a vertex algebra by $\{a_i(z) \mid i \in I\}$.*

As shown in [LI], there is a similar reconstruction property for kernels of surjective morphisms in \mathcal{R} . Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a morphism in \mathcal{R} with kernel \mathcal{J} , such that f maps $\mathcal{A}_{(k)}$ onto $\mathcal{B}_{(k)}$ for all $k \geq 0$. The kernel J of the induced map $gr(f) : gr(\mathcal{A}) \rightarrow gr(\mathcal{B})$ is a homogeneous ∂ -ideal (i.e., $\partial J \subset J$). A set $\{a_i \mid i \in I\}$ such that a_i is homogeneous of degree d_i is said to generate J as a ∂ -ideal if $\{\partial^k a_i \mid i \in I, k \geq 0\}$ generates J as an ideal.

Lemma 3.2. *Let $\{a_i \mid i \in I\}$ be a generating set for J as a ∂ -ideal, where a_i is homogeneous of degree d_i . Then there exist vertex operators $a_i(z) \in \mathcal{A}_{(d_i)}$ with $\phi_{d_i}(a_i(z)) = a_i$, such that $\{a_i(z) \mid i \in I\}$ generates \mathcal{J} as a vertex algebra ideal.*

For any Lie algebra \mathfrak{g} and bilinear form B , $V_k(\mathfrak{g}, B)$ admits a good increasing filtration

$$(3.5) \quad V_k(\mathfrak{g}, B)_{(0)} \subset V_k(\mathfrak{g}, B)_{(1)} \subset \cdots, \quad V_k(\mathfrak{g}, B) = \bigcup_{j \geq 0} V_k(\mathfrak{g}, B)_{(j)},$$

where $V_k(\mathfrak{g}, B)_{(j)}$ is defined to be the vector space spanned by iterated Wick products of the generators X^{ξ_i} and their derivatives, of length at most j . We say that elements of $V_k(\mathfrak{g}, B)_{(j)} \setminus V_k(\mathfrak{g}, B)_{(j-1)}$ have degree j . For any group G of weight-preserving automorphisms of $V_k(\mathfrak{g}, B)$, this filtration is G -invariant, and we have an isomorphism of commutative algebras

$$(3.6) \quad gr(V_k(\mathfrak{g}, B)) \cong Sym \bigoplus_{j \geq 0} V_j,$$

where $V_j \cong \mathfrak{g}$ as G -modules. The generators of $gr(V_k(\mathfrak{g}, B))$ are $X_j^{\xi_i}$, which correspond to the vertex operators $\partial^j X^{\xi_i}$. Since $V_k(\mathfrak{g}, B)$ has a basis consisting of iterated Wick products of the generators and their derivatives, $V_k(\mathfrak{g}, B) \cong gr(V_k(\mathfrak{g}, B))$ as vector spaces, although not canonically. The filtration (3.5) is inherited by $V_k(\mathfrak{g}, B)^G$, and

$$(3.7) \quad gr(V_k(\mathfrak{g}, B))^G \cong gr(V_k(\mathfrak{g}, B)^G) \cong (Sym \bigoplus_{j \geq 0} V_j)^G$$

as commutative rings. Finally, note that $Sym \bigoplus_{j \geq 0} V_j$ has a natural ∂ -ring structure, where the derivation ∂ acts on the generators $x_j \in V_j$ by $\partial(x_j) = x_{j+1}$. Clearly ∂ commutes with the action of G , so ∂ acts on $(Sym \bigoplus_{j \geq 0} V_j)^G$, and (3.7) is an isomorphism of ∂ -rings.

4. SOME CLASSICAL INVARIANT THEORY

We briefly recall some terminology and results about invariant rings of the form

$$R = (Sym \bigoplus_{j \geq 0} V_j)^G,$$

where each V_j is isomorphic to some fixed G -module V . In the terminology of Weyl, a *first fundamental theorem of invariant theory* for the pair (G, V) is a set of generators for R . In some treatments, a first fundamental theorem is defined as a set of generators for the larger ring $(Sym \bigoplus_{j \geq 0} (V_j \oplus V_j^*))^G$, where $V_j^* \cong V^*$ as G -modules, but here we only need to consider $(Sym \bigoplus_{j \geq 0} V_j)^G$. A *second fundamental theorem of invariant theory* for (G, V) is a set of generators for the ideal of relations among the generators of R .

First and second fundamental theorems of invariant theory are known for the standard representations of the classical groups [We] and for the adjoint representations of the classical groups [P], but in general it is quite difficult to describe these rings explicitly. However, there is a certain finiteness theorem of Weyl (II.5 Theorem 2.5A of [We]) that will be useful. Note that for all $p \geq 0$, there is an action of GL_p on $\bigoplus_{j=0}^{p-1} V_j$ which commutes with the action of G . The inclusions $GL_p \hookrightarrow GL_q$ for $p < q$ sending

$$M \rightarrow \begin{bmatrix} M & 0 \\ 0 & I_{q-p} \end{bmatrix}$$

induce an action of $GL_\infty = \lim_{p \rightarrow \infty} GL_p$ on $\bigoplus_{j \geq 0} V_j$. We obtain an action of GL_∞ on $Sym \bigoplus_{j \geq 0} V_j$ by algebra automorphisms, which commutes with the action of G . Hence GL_∞ acts on R as well. The elements $\sigma \in GL_\infty$ are known as *polarization operators*, and given $f \in R$, σf is known as a polarization of f .

Theorem 4.1. (Weyl) *R is generated by the set of polarizations of any set of generators for $(Sym \bigoplus_{j=0}^{n-1} V_j)^G$, where $n = \dim(V)$. Since G is reductive, $(Sym \bigoplus_{j=0}^{n-1} V_j)^G$ is finitely generated. Hence there exists a finite set $\{f_1, \dots, f_r\}$ of homogeneous elements of R , whose polarizations generate R .*

Note that taking the complete set of polarizations is unnecessary. We can always find a *minimal* set S of polarizations of $\{f_1, \dots, f_r\}$ which generates R , such that no element of S is a nontrivial polynomial in other elements of S . Clearly for all $p \geq n$, $S \cap (Sym \bigoplus_{j=0}^p V_j)^G$ will be a finite generating set for $(Sym \bigoplus_{j=0}^p V_j)^G$. Next, recall that R has the structure of a ∂ -ring. For our purposes, we will need a minimal set of generators for R as a ∂ -ring, and our set S may not be minimal in this sense. Any element of S of the form $\partial^k s$ for $s \in S$ and $k > 0$, and any nontrivial polynomial in the elements of S and their derivatives can be removed, and the resulting set will still generate R as a ∂ -ring.

Let \mathcal{V} be a vertex algebra with a good increasing filtration such that $\mathcal{V}^G \cong gr(\mathcal{V}^G)$ as linear spaces, and $gr(\mathcal{V})^G \cong gr(\mathcal{V}^G) \cong (Sym \bigoplus_{j \geq 0} V_j)^G = R$ as ∂ -rings, where V_j is isomorphic to some finite-dimensional G -module V , for all $j \geq 0$. Our main example of such a vertex algebra is $V_k(\mathfrak{g}, B)$ where $V \cong \mathfrak{g}$. Choose homogeneous elements $\{f_1, \dots, f_r\} \subset R$

whose polarizations generate R , and choose a minimal set S of polarizations which generates R as a ∂ -ring. Let $T = \{s(z) \mid s \in S\} \subset \mathcal{V}^G$ be the set of vertex operators corresponding to S under the linear isomorphism $\mathcal{V}^G \cong R$. By Lemma 3.1, T is a strong generating set for \mathcal{V}^G .

Given a homogeneous polynomial $p \in \text{gr}(\mathcal{V}^G) \cong R$ of degree d , a *normal ordering* of p will be a choice of normally ordered polynomial $P \in (\mathcal{V}^G)_{(d)}$, obtained by replacing each $s \in S$ by $s(z) \in T$, and replacing ordinary products with iterated Wick products of the form (2.2). Of course P is not unique, but for any choice of P we have $\phi_d(P) = p$, where

$$\phi_d : (\mathcal{V}^G)_{(d)} \rightarrow (\mathcal{V}^G)_{(d)} / (\mathcal{V}^G)_{(d-1)} \subset \text{gr}(\mathcal{V}^G)$$

is the usual projection.

Suppose that p is a relation among the generators of R coming from the second fundamental theorem for (G, V) , which we may assume to be homogeneous of degree d . Let $P^d \in \mathcal{V}^G$ be some normal ordering of p . Since $\text{gr}(\mathcal{V}^G) \cong R$ as graded rings, it follows that P^d lies in $\mathcal{V}_{(d-1)}^G$. The polynomial $\phi_{d-1}(P^d) \in R$ is homogeneous of degree $d-1$; if it is nonzero, it can be expressed as a polynomial in the variables $s \in S$ and their derivatives. Choose some normal ordering of this polynomial, and call this vertex operator $-P^{d-1}$. Then $P^d + P^{d-1}$ has the property that

$$\phi_d(P^d + P^{d-1}) = p, \quad P^d + P^{d-1} \in (\mathcal{V}^G)_{(d-2)}.$$

Continuing this process, we arrive at a vertex operator

$$(4.1) \quad P = \sum_{k=1}^d P^k \in \mathcal{V}^G$$

which is identically zero. We view P as a quantum correction of the relation p , and it is easy to see by induction on degree that all normally ordered polynomial relations in \mathcal{V}^G among the elements of T and their derivatives, are consequences of relations of this kind.

In general, R will not be finitely generated as a ∂ -ring. However, since the relations (4.1) are more complicated than their classical counterparts, it is still possible for \mathcal{V}^G to be strongly generated as a vertex algebra by a finite subset $T' \subset T$. For this to happen, each element $t \in T \setminus T'$ must admit a “decoupling relation” expressing it as a normally ordered polynomial in the elements of T' and their derivatives. Given a relation in \mathcal{V}^G of the form (4.1), suppose that some $t \in T \setminus T'$ appears in P^k for some $k < d$, with nonzero coefficient. If the remaining terms in (4.1) only depend on the elements of T' and their derivatives, we can solve for t to obtain such a decoupling relation. The existence of a complete set of decoupling relations for all $t \in T \setminus T'$ is a subtle and nonclassical phenomenon, since it depends on precise properties of the quantum corrections appearing in relations of the form (4.1).

5. INVARIANT SUBALGEBRAS OF HEISENBERG VERTEX ALGEBRAS

In [LIII] we studied invariant subalgebras of the rank n Heisenberg vertex algebra $\mathcal{H}(n)$, and we briefly recall our main results. Recall that $\mathcal{H}(n) \cong \text{gr}(\mathcal{H}(n))$ as linear spaces, and $\text{gr}(\mathcal{H}(n)) \cong \text{Sym} \bigoplus_{j \geq 0} V_j$ as commutative rings. Here V_j is spanned by $\{\alpha_j^i \mid i = 1, \dots, n\}$, where α_j^i is the image of $\partial^j \alpha^i(z)$ under the projection $\phi_1 : \mathcal{H}(n)_{(1)} \rightarrow \mathcal{H}(n)_{(1)} / \mathcal{H}(n)_{(0)} \subset \text{gr}(\mathcal{H}(n))$. The action of $O(n)$ on $\mathcal{H}(n)$ preserves the filtration and induces an action of

$O(n)$ on $gr(\mathcal{H}(n))$, and $V_j \cong \mathbb{C}^n$ as $O(n)$ -modules for all $j \geq 0$. We have a linear isomorphism $\mathcal{H}(n)^{O(n)} \cong gr(\mathcal{H}(n)^{O(n)})$, and isomorphisms of commutative rings

$$(5.1) \quad gr(\mathcal{H}(n)^{O(n)}) \cong (gr(\mathcal{H}(n)))^{O(n)} \cong (Sym \bigoplus_{j \geq 0} V_j)^{O(n)}.$$

The following classical theorem of Weyl [We] describes the generators and relations of the ring $(Sym \bigoplus_{j \geq 0} V_j)^{O(n)}$:

Theorem 5.1. *For $j \geq 0$, let V_j be the copy of the standard $O(n)$ -module \mathbb{C}^n with orthonormal basis $\{x_{i,j} \mid i = 1, \dots, n\}$. The invariant ring $(Sym \bigoplus_{j \geq 0} V_j)^{O(n)}$ is generated by the quadratics*

$$(5.2) \quad q_{a,b} = \sum_{i=1}^n x_{i,a} x_{i,b}, \quad 0 \leq a \leq b.$$

For $a > b$, define $q_{a,b} = q_{b,a}$, and let $\{Q_{a,b} \mid a, b \geq 0\}$ be commuting indeterminates satisfying $Q_{a,b} = Q_{b,a}$ and no other algebraic relations. The kernel I_n of the homomorphism $\mathbb{C}[Q_{a,b}] \rightarrow (Sym \bigoplus_{j \geq 0} V_j)^{O(n)}$ sending $Q_{a,b} \mapsto q_{a,b}$ is generated by the $(n+1) \times (n+1)$ determinants

$$(5.3) \quad d_{I,J} = \begin{bmatrix} Q_{i_0,j_0} & \cdots & Q_{i_0,j_n} \\ \vdots & & \vdots \\ Q_{i_n,j_0} & \cdots & Q_{i_n,j_n} \end{bmatrix}.$$

In this notation, $I = (i_0, \dots, i_n)$ and $J = (j_0, \dots, j_n)$ are lists of integers satisfying

$$(5.4) \quad 0 \leq i_0 < \cdots < i_n, \quad 0 \leq j_0 < \cdots < j_n.$$

Under the projection

$$\phi_2 : (\mathcal{H}(n)^{O(n)})_{(2)} \rightarrow (\mathcal{H}(n)^{O(n)})_{(2)} / (\mathcal{H}(n)^{O(n)})_{(1)} \subset gr(\mathcal{H}(n)^{O(n)}) \cong (Sym \bigoplus_{j \geq 0} V_j)^{O(n)},$$

the generators $q_{a,b}$ of $(Sym \bigoplus_{j \geq 0} V_j)^{O(n)}$ correspond to vertex operators $\omega_{a,b}$ given by

$$(5.5) \quad \omega_{a,b} = \sum_{i=1}^n : \partial^a \alpha^i \partial^b \alpha^i :, \quad 0 \leq a \leq b.$$

By Lemma 3.1, the set $\{\omega_{a,b} \mid 0 \leq a \leq b\}$ is a strong generating set for $\mathcal{H}(n)^{O(n)}$. In fact, there is a somewhat more economical set of strong generators which is more natural from the point of view of vertex algebras. For each $m \geq 0$, let A_m denote the vector space spanned by $\{\omega_{a,b} \mid a+b = m\}$, which is homogeneous of weight $m+2$. We have $\dim(A_{2m}) = m+1 = \dim(A_{2m+1})$ and

$$\dim(A_{2m} / \partial(A_{2m-1})) = 1, \quad \dim(A_{2m+1} / \partial(A_{2m})) = 0.$$

For $m \geq 0$, define

$$(5.6) \quad j^{2m} = \omega_{0,2m}.$$

Clearly j^{2m} is not a total derivative, so A_{2m} has a decomposition

$$(5.7) \quad A_{2m} = \partial(A_{2m-1}) \oplus \langle j^{2m} \rangle = \partial^2(A_{2m-2}) \oplus \langle j^{2m} \rangle,$$

where $\langle j^{2m} \rangle$ is the linear span of j^{2m} . Similarly,

$$(5.8) \quad A_{2m+1} = \partial^2(A_{2m-1}) \oplus \langle \partial j^{2m} \rangle = \partial^3(A_{2m-2}) \oplus \langle \partial j^{2m} \rangle,$$

and it is easy to see that $\{\partial^{2i} j^{2m-2i} \mid 0 \leq i \leq m\}$ and $\{\partial^{2i+1} j^{2m-2i} \mid 0 \leq i \leq m\}$ are bases of A_{2m} and A_{2m+1} , respectively. It follows that $\{j^{2m} \mid m \geq 0\}$ is an alternative strong generating set for $\mathcal{H}(n)^{O(n)}$.

We can realize $\mathcal{H}(n)^{O(n)}$ as the quotient of a vertex algebra \mathcal{V}_n which is *freely* generated by vertex operators $\{J^{2m}(z) \mid m \geq 0\}$, under a map $\pi_n : \mathcal{V}_n \rightarrow \mathcal{H}(n)^{O(n)}$ sending $J^{2m} \mapsto j^{2m}$. In other words, there are no normally ordered polynomial relations in \mathcal{V}_n among the generators and their derivatives. There is an alternative strong generating set

$$\{\Omega_{a,b} \mid 0 \leq a \leq b\}$$

for \mathcal{V}_n such that $\pi_n(\Omega_{a,b}) = \omega_{a,b}$. Recall the variables $Q_{a,b}$ and $q_{a,b}$ appearing in Theorem 5.1. Since \mathcal{V}_n is freely generated by $\{J^{2m} \mid m \geq 0\}$, and the sets $\{\Omega_{a,b} \mid 0 \leq a \leq b\}$ and $\{\partial^k J^{2m} \mid k, m \geq 0\}$ form bases for the same vector space, we may identify $gr(\mathcal{V}_n)$ with $\mathbb{C}[Q_{a,b}]$, and we identify $gr(\mathcal{H}(n)^{O(n)})$ with $\mathbb{C}[q_{a,b}]/I_n$. Under this identification, $gr(\pi_n) : gr(\mathcal{V}_n) \rightarrow gr(\mathcal{H}(n)^{O(n)})$ is just the quotient map sending $Q_{a,b} \mapsto q_{a,b}$.

There is a good increasing filtration on \mathcal{V}_n such that $(\mathcal{V}_n)_{(2k)}$ is spanned by iterated Wick products of the generators $\Omega_{a,b}$, of length at most k , and $(\mathcal{V}_n)_{(2k+1)} = (\mathcal{V}_n)_{(2k)}$. Equipped with this filtration, \mathcal{V}_n lies in the category \mathcal{R} , and π_n is a morphism in \mathcal{R} . Clearly π_n maps each filtered piece $(\mathcal{V}_n)_{(k)}$ onto $(\mathcal{H}(n)^{O(n)})_{(k)}$, so the hypotheses of Lemma 3.2 are satisfied. Since $I_n = Ker(gr(\pi_n))$ is generated by the determinants $d_{I,J}$, we can apply Lemma 3.2 to find vertex operators

$$(5.9) \quad D_{I,J} \in (\mathcal{V}_n)_{(2n+2)}$$

satisfying $\phi_{2n+2}(D_{I,J}) = d_{I,J}$, such that $\{D_{I,J}\}$ generates \mathcal{I}_n . Here

$$\phi_{2n+2} : (\mathcal{V}_n)_{(2n+2)} \rightarrow (\mathcal{V}_n)_{(2n+2)}/(\mathcal{V}_n)_{(2n+1)} \subset gr(\mathcal{V}_n)$$

is the usual projection. Since $\Omega_{a,b}$ has weight $a + b + 2$, it follows that

$$wt(D_{I,J}) = |I| + |J| + 2n + 2, \quad |I| = \sum_{a=0}^n i_a, \quad |J| = \sum_{a=0}^n j_a.$$

Since there are no relations in $\mathcal{H}(n)^{O(n)}$ of degree less than $2n + 2$, $D_{I,J}$ is uniquely determined by the conditions

$$(5.10) \quad \phi_{2n+2}(D_{I,J}) = d_{I,J}, \quad \pi_n(D_{I,J}) = 0.$$

Given a homogeneous polynomial $p \in gr(\mathcal{V}_n) \cong \mathbb{C}[Q_{a,b}]$ of degree k in the variables $Q_{a,b}$, recall that a *normal ordering* of p is a choice of normally ordered polynomial $P \in (\mathcal{V}_n)_{(2k)}$, obtained by replacing $Q_{a,b}$ by $\Omega_{a,b}$, and replacing ordinary products with iterated Wick products. For any choice of P we have $\phi_{2k}(P) = p$. Let $D_{I,J}^{2n+2} \in (\mathcal{V}_n)_{(2n+2)}$ be some normal ordering of $d_{I,J}$. Then $\pi_n(D_{I,J}^{2n+2}) \in (\mathcal{H}(n)^{O(n)})_{(2n)}$. Using the procedure outlined in Section 4, we can find a sequence of quantum corrections $D_{I,J}^{2k}$ for $k = 1, \dots, n$, which are homogeneous, normally ordered polynomials of degree k in the variables $\Omega_{a,b}$, such that $\sum_{k=1}^{n+1} D_{I,J}^{2k}$ lies in the kernel of π_n . We have

$$(5.11) \quad D_{I,J} = \sum_{k=1}^{n+1} D_{I,J}^{2k},$$

since $D_{I,J}$ is uniquely characterized by (5.10).

In this decomposition, the term $D_{I,J}^2$ lies in the space A_m spanned by $\{\Omega_{a,b} \mid a+b=m\}$, for $m = |I| + |J| + 2n$. From now on, we will restrict ourselves to the case where m is even. In this case, $A_m = \partial^2(A_{m-2}) \oplus \langle J^m \rangle$, and we define $pr_m : A_m \rightarrow \langle J^m \rangle$ to be the projection onto the second term. In [LIII] we defined the *remainder* $R_{I,J}$ to be $pr_m(D_{I,J}^2)$. We write

$$(5.12) \quad R_{I,J} = R_n(I, J)J^m,$$

so that $R_n(I, J)$ denotes the coefficient of J^m in $pr_m(D_{I,J}^2)$. By Lemma 4.4 of [LIII], $R_n(I, J)$ is independent of the choice of decomposition (5.11).

Define a grading $\mathcal{H}(n) = \bigoplus_{r \geq 0} \mathcal{H}(n)^{(r)}$, where $\mathcal{H}(n)^{(r)}$ is spanned by all normally ordered monomials

$$: \partial^{k_1^1} \alpha_1 \dots \partial^{k_{r_1}^1} \alpha_1 \partial^{k_1^2} \alpha_2 \dots \partial^{k_{r_2}^2} \alpha_2 \dots \partial^{k_1^n} \alpha^n \dots \partial^{k_{r_n}^n} \alpha^n :.$$

In this notation, $(k_1^i, \dots, k_{r_i}^i)$ are sequences of integers satisfying $0 \leq k_1^i \leq k_2^i \leq \dots \leq k_{r_i}^i$ for all $i = 1, \dots, n$, and

$$\sum_{i=1}^n k_1^i + \dots + k_{r_i}^i = r.$$

This grading is compatible with the good increasing filtration on $\mathcal{H}(n)$ in the sense that $\mathcal{H}(n)_{(r)} = \bigoplus_{j=0}^r \mathcal{H}(n)^{(j)}$ and $\mathcal{H}(n)^{(r)} \cong \mathcal{H}(n)_{(r)} / \mathcal{H}(n)_{(r-1)}$. Moreover, the action of $O(n)$ preserves the grading, so we obtain a grading $\mathcal{H}(n)^{O(n)} = \bigoplus_{r \geq 0} (\mathcal{H}(n)^{(r)})^{O(n)}$ and an isomorphism of graded vector spaces

$$i_n : \mathcal{H}(n)^{O(n)} \rightarrow (\text{Sym} \bigoplus_{j \geq 0} V_j)^{O(n)}, \quad V_j \cong \mathbb{C}^n.$$

Let $p \in (\text{Sym} \bigoplus_{j \geq 0} V_j)^{O(n)}$ be a homogeneous polynomial of degree $2d$, and let $f = (i_n)^{-1}(p) \in (\mathcal{H}(n)^{O(n)})^{(2d)}$ be the corresponding homogeneous vertex operator. Let $F \in (\mathcal{V}_n)_{(2d)}$ be a vertex operator satisfying $\pi_n(F) = f$, where $\pi_n : \mathcal{V}_n \rightarrow \mathcal{H}(n)^{O(n)}$ is the projection. We can write $F = \sum_{k=1}^d F^{2k}$, where F^{2k} is a normally ordered polynomial of degree k in the vertex operators $\Omega_{a,b}$.

Next, let \tilde{V} be the vector space \mathbb{C}^{n+1} , and let

$$\tilde{q}_{a,b} \in (\text{Sym} \bigoplus_{j \geq 0} \tilde{V}_j)^{O(n+1)}$$

be the generator given by (5.2). Here \tilde{V}_j is isomorphic to \tilde{V} for $j \geq 0$. Let \tilde{p} be the polynomial of degree $2d$ obtained from p by replacing each $q_{a,b}$ with $\tilde{q}_{a,b}$, and let $\tilde{f} = (i_{n+1})^{-1}(\tilde{p}) \in (\mathcal{H}(n+1)^{O(n+1)})^{(2d)}$ be the corresponding homogeneous vertex operator. Finally, let $\tilde{F}^{2k} \in \mathcal{V}_{n+1}$ be the vertex operator obtained from F^{2k} by replacing each $\Omega_{a,b}$ with the corresponding vertex operator $\tilde{\Omega}_{a,b} \in \mathcal{V}_{n+1}$, and let $\tilde{F} = \sum_{k=1}^d \tilde{F}^{2k}$.

Lemma 5.2. *Fix $n \geq 1$, and let $D_{I,J}$ be a relation in \mathcal{V}_n of the form (5.9). There exists a decomposition $D_{I,J} = \sum_{k=1}^{n+1} D_{I,J}^{2k}$ of the form (5.11) such that the corresponding vertex operator*

$$\tilde{D}_{I,J} = \sum_{k=1}^{n+1} \tilde{D}_{I,J}^{2k} \in \mathcal{V}_{n+1}$$

has the property that $\pi_{n+1}(\tilde{D}_{I,J})$ lies in the homogeneous subspace $(\mathcal{H}(n+1)^{O(n+1)})^{(2n+2)}$ of degree $2n+2$.

Proof. The argument is the same as the proof of the corresponding statement (Corollary 4.14 of [LI]) and is omitted. \square

We shall find a recursive formula for $R_n(I, J)$ in the case where $m = |I| + |J| + 2n$ is even. Note that for $a + b = m$, we have $pr_m(\Omega_{a,b}) = (-1)^m J^m$. Using this fact, it is easy to check that for $n = 1$, $I = (i_0, i_1)$ and $J = (j_0, j_1)$,

$$(5.13) \quad R_1(I, J) = (-1)^{j_0} \left(\frac{(-1)^{i_0}}{2 + i_0 + i_1} + \frac{(-1)^{j_1}}{2 + i_1 + j_1} \right) - (-1)^{j_1} \left(\frac{(-1)^{i_0}}{2 + i_0 + i_1} + \frac{(-1)^{j_0}}{2 + i_1 + j_0} \right) + \\ (-1)^{i_1} \left(\frac{(-1)^{i_0}}{2 + i_0 + j_0} + \frac{(-1)^{j_1}}{2 + j_0 + j_1} \right) - (-1)^{i_1} \left(\frac{(-1)^{i_0}}{2 + i_0 + j_1} + \frac{(-1)^{j_0}}{2 + j_0 + j_1} \right).$$

Next, we will express $R_n(I, J)$ for all $I = (i_0, \dots, i_n)$, $J = (j_0, \dots, j_n)$ in terms of the expressions $R_{n-1}(K, L)$. Let $d_{I,J}$ and $D_{I,J}$ be the corresponding elements of $\mathbb{C}[Q_{j,k}]$ and \mathcal{V}_n , respectively. By expanding the determinant $d_{I,J}$ along its first column, we have

$$d_{I,J} = \sum_{r=0}^n (-1)^r Q_{i_r, j_0} d_{I_r, J'},$$

where $I_r = (i_0, \dots, \widehat{i_r}, \dots, i_n)$ is obtained from I by omitting i_r , and $J' = (j_1, \dots, j_n)$ is obtained from J by omitting j_0 . Let $D_{I_r, J'} \in \mathcal{V}_{n-1}$ be the vertex operator corresponding to the $n \times n$ determinant $d_{I_r, J'}$. By Lemma 5.2, there exists a decomposition

$$D_{I_r, J'} = \sum_{i=1}^n D_{I_r, J'}^{2i}$$

such that the corresponding element $\tilde{D}_{I_r, J'} = \sum_{i=1}^n \tilde{D}_{I_r, J'}^{2i} \in \mathcal{V}_n$ has the property that $\pi_n(\tilde{D}_{I_r, J'})$ lies in the homogeneous subspace $(\mathcal{H}(n)^{O(n)})^{(2n)}$ of degree $2n$. We have

$$(5.14) \quad \sum_{r=0}^n (-1)^r : \Omega_{i_r, j_0} \tilde{D}_{I_r, J'} : = \sum_{r=0}^n \sum_{i=1}^n (-1)^r : \Omega_{i_r, j_0} \tilde{D}_{I_r, J'}^{2i} : .$$

The right hand side of (5.14) consists of normally ordered monomials of degree at least 2 in the vertex operators $\Omega_{a,b}$, and hence contributes nothing to the remainder $R_{I,J}$. Since $\pi_n(\tilde{D}_{I_r, J'})$ is homogeneous of degree $2n$, $\pi_n(: \Omega_{i_r, j_0} \tilde{D}_{I_r, J'} :)$ consists of a piece of degree $2n + 2$ and a piece of degree $2n$ coming from all double contractions of Ω_{i_r, j_0} with terms in $\tilde{D}_{I_r, J'}$, which lower the degree by two. The component of

$$\pi_n \left(\sum_{r=0}^n (-1)^r : \Omega_{i_r, j_0} \tilde{D}_{I_r, J'} : \right) \in \mathcal{H}(n)^{O(n)}$$

in degree $2n + 2$ must cancel since this sum corresponds to the classical determinant $d_{I,J}$. The component of $: \Omega_{i_r, j_0} \tilde{D}_{I_r, J'} :$ in degree $2n$ is

$$S_r = (-1)^{i_r} \left(\sum_k \frac{\tilde{D}_{I_r, k, J'}}{i_k + i_r + 2} + \sum_l \frac{\tilde{D}_{I_r, J'_l}}{j_l + i_r + 2} \right) + (-1)^{j_0} \left(\sum_k \frac{\tilde{D}_{I_r, k, J'}}{i_k + j_0 + 2} + \sum_l \frac{\tilde{D}_{I_r, J'_l}}{j_l + j_0 + 2} \right).$$

In this notation, for $k = 0, \dots, n$ and $k \neq r$, $I_{r,k}$ is obtained from $I_r = (i_0, \dots, \widehat{i_r}, \dots, i_n)$ by replacing the entry i_k with $i_k + i_r + j_0 + 2$. Similarly, for $l = 1, \dots, n$, J'_l is obtained from

$J' = (j_1, \dots, j_n)$ by replacing j_l with $j_l + i_r + j_0 + 2$. It follows that

$$(5.15) \quad \pi_n \left(\sum_{r=0}^n (-1)^r : \Omega_{i_r, j_0} \tilde{D}_{I_r, J'} : \right) = \pi_n \left(\sum_{r=0}^n (-1)^r S_r \right).$$

Combining (5.14) and (5.15), we can regard

$$\sum_{r=0}^n \sum_{i=1}^n (-1)^r : \Omega_{i_r, j_0} \tilde{D}_{I_r, J'}^{2i} : - \sum_{i=0}^n (-1)^r S_r$$

as a decomposition of $D_{I, J}$ of the form $D_{I, J} = \sum_{k=1}^{n+1} D_{I, J}^{2k}$ where the leading term $D_{I, J}^{2n+2} = \sum_{r=0}^n (-1)^r : \Omega_{i_r, j_0} \tilde{D}_{I_r, J'}^{2n} :$. It follows that $R_n(I, J)$ is the negative of sum of the terms $R_{n-1}(K, L)$ corresponding to each $\tilde{D}_{K, L}$ appearing in $\sum_{r=0}^n (-1)^r S_r$. We therefore obtain the following recursive formula:

$$(5.16) \quad R_n(I, J) = - \sum_{r=0}^n (-1)^r (-1)^{i_r} \left(\sum_k \frac{R_{n-1}(I_{r, k}, J')}{i_k + i_r + 2} + \sum_l \frac{R_{n-1}(I_r, J'_l)}{j_l + i_r + 2} \right) \\ - \sum_{r=0}^n (-1)^r (-1)^{j_0} \left(\sum_k \frac{R_{n-1}(I_{r, k}, J')}{i_k + j_0 + 2} + \sum_l \frac{R_{n-1}(I_r, J'_l)}{j_l + j_0 + 2} \right).$$

For $I = (0, 1, \dots, n) = J$, $D_{I, J}$ is the unique element (up to scalar multiples) of minimal weight $n^2 + 3n + 2$ in the ideal $\mathcal{I}_n \subset \mathcal{V}_n$. Let R_n denote the coefficient $R_n(I, J)$ of J^{n^2+3n} appearing in the remainder $R_{I, J}$. Using our recursive formula we can calculate the first few values of R_n :

TABLE 1. Some values of R_n

n	1	2	3	4	5	6
R_n	$\frac{5}{4}$	$\frac{149}{600}$	$-\frac{2419}{705600}$	$-\frac{67619}{18670176000}$	$\frac{1391081}{4879637199360000}$	$\frac{40984649}{25145492674607585280000}$

In [LIII], the values R_1 , R_2 , and R_3 also appear, but R_1 was stated (erroneously) to be $-\frac{5}{4}$. The condition $R_n \neq 0$ gives us a decoupling relation

$$j^{n^2+3n} = P(j^0, j^2, \dots, j^{n^2+3n-2})$$

in $\mathcal{H}(n)^{O(n)}$, where P is a normally ordered polynomial in $j^0, j^2, \dots, j^{n^2+3n-2}$ and their derivatives. By Theorem 4.7 of [LIII], if such a decoupling relation exists, by applying the operator $j^2 \circ_1$ repeatedly, we can construct higher decoupling relations

$$j^{2r} = Q_{2r}(j^0, j^2, \dots, j^{n^2+3n-2})$$

for all $r > \frac{1}{2}(n^2 + 3n)$. Since $R_n \neq 0$ for $n \leq 6$, these calculations imply the following result:

Theorem 5.3. *For $n \leq 6$, $\mathcal{H}(n)^{O(n)}$ has a minimal strong generating set $\{j^0, j^2, \dots, j^{n^2+3n-2}\}$, and in particular is a \mathcal{W} -algebra of type $\mathcal{W}(2, 4, \dots, n^2 + 3n)$.*

Even though we do not prove the conjecture that $R_n \neq 0$ for all n , we can still show that $\mathcal{H}(n)^{O(n)}$ is strongly finitely generated. Let

$$I = (0, 1, \dots, n), \quad J = (0, 1, \dots, n-1, a), \quad a \geq n, \quad a \equiv n \pmod{2},$$

and define a function $f(a) = R_n(I, J)$. It is immediate from (5.13) and (5.16) that $f(a)$ is a rational function of a , so it has at most finitely many zeroes. Let a_0 be the minimal value of a such that $f(a_0) \neq 0$, and let $m = \frac{1}{2}(n^2 + 2n + a_0)$. It follows that the coefficient of J^{2m} appearing in $D_{I,J}^2$ is nonzero, so we obtain a decoupling relation

$$j^{2m} = P(j^0, j^2, \dots, j^{2m-2}).$$

As in the proof of Theorem 4.7 of [LIII], applying the operator $j^2 \circ_1$ repeatedly yields decoupling relations $j^{2r} = Q_r(j^0, j^2, \dots, j^{2m-2})$ for all $r > m$. This proves

Theorem 5.4. *For all $n \geq 1$, $\mathcal{H}(n)^{O(n)}$ is strongly generated by $\{j^0, j^2, \dots, j^{2m-2}\}$.*

For an arbitrary reductive group G of automorphisms of $\mathcal{H}(n)$, the structure of $\mathcal{H}(n)^G$ can be understood by decomposing $\mathcal{H}(n)^G$ as a module over $\mathcal{H}(n)^{O(n)}$. By Theorem 5.1 of [LIII], the Zhu algebra of $\mathcal{H}(n)^{O(n)}$ is abelian, which implies that all its irreducible, admissible modules are highest-weight modules. Moreover, both $\mathcal{H}(n)$ and $\mathcal{H}(n)^G$ decompose as direct sums of irreducible, highest-weight $\mathcal{H}(n)^{O(n)}$ -modules. By Theorem 6.1 of [LIII], there is a finite set of irreducible $\mathcal{H}(n)^{O(n)}$ -submodules of $\mathcal{H}(n)^G$ whose direct sum contains an (infinite) strong generating for $\mathcal{H}(n)^G$. This shows that $\mathcal{H}(n)^G$ is finitely generated as a vertex algebra. If we assume the conjecture that $R_n \neq 0$, so that $\{j^0, j^2, \dots, j^{n^2+3n-2}\}$ strongly generates $\mathcal{H}(n)^{O(n)}$, Theorem 6.9 of [LIII] shows that $\mathcal{H}(n)^G$ is strongly finitely generated. In fact, the only step where this conjecture was needed was the proof of Lemma 6.7 of [LIII], which gives a finiteness property of each irreducible, highest-weight $\mathcal{H}(n)^{O(n)}$ -submodule of $\mathcal{H}(n)$. This proof of this lemma goes through if we replace $\frac{1}{2}(n^2 + 3n)$ with the integer m appearing in Theorem 5.4 above. This proves

Theorem 5.5. *For all $n \geq 1$ and any reductive group $G \subset O(n)$, $\mathcal{H}(n)^G$ is strongly finitely generated.*

6. INVARIANT SUBALGEBRAS OF AFFINE VERTEX ALGEBRAS

Let \mathfrak{g} be an n -dimensional Lie algebra equipped with a nondegenerate, symmetric, invariant bilinear form B , and let G be a reductive group of automorphisms of $V_k(\mathfrak{g}, B)$ preserving the conformal weight grading, for all $k \in \mathbb{C}$. In this section, we will prove Theorem 1.1, which states that $V_k(\mathfrak{g}, B)^G$ is strongly finitely generated for generic values of k . Since G acts on \mathfrak{g} and preserves both the bracket and the bilinear form, G lies in $O(n)$, and therefore acts on $\mathcal{H}(n)$ as well. The key idea behind the proof is that both $V_k(\mathfrak{g}, B)$ and $\mathcal{H}(n)$ admit G -invariant, good increasing filtrations, and we have linear isomorphisms

$$\mathcal{H}(n)^G \cong \text{gr}(\mathcal{H}(n)^G) \cong \text{gr}(\mathcal{H}(n))^G \cong \text{gr}(V_k(\mathfrak{g}, B)^G) \cong \text{gr}(V_k(\mathfrak{g}, B))^G \cong V_k(\mathfrak{g}, B)^G$$

and isomorphisms of graded commutative rings

$$\text{gr}(V_k(\mathfrak{g}, B))^G \cong (\text{Sym} \bigoplus_{j \geq 0} V_j)^G \cong \text{gr}(\mathcal{H}(n))^G,$$

where $V_j \cong \mathbb{C}^n \cong \mathfrak{g}$ as G -modules, for all $j \geq 0$. Hence both $V_k(\mathfrak{g}, B)^G$ and $\mathcal{H}(n)^G$ can be viewed as deformations of the same classical invariant ring $R = (\text{Sym} \bigoplus_{j \geq 0} V_j)^G$. By Theorem 4.1, there is a finite set $\{f_1, \dots, f_t\}$ of homogeneous generators for $(\text{Sym} \bigoplus_{j=0}^{n-1} V_j)^G$, whose polarizations generate R . Let $\{d_1, \dots, d_r\}$ be the set of distinct degrees of the elements $\{f_1, \dots, f_t\}$, with $d_1 < d_2 < \dots < d_r$. Since the action of GL_∞ on R preserves degree, all polarizations of these elements must have degree d_j for some $j = 1, \dots, r$. As in Section 4, we may choose a minimal subset S of these polarizations, which generates R as a ∂ -ring. It follows from Lemma 3.1 that the sets $T \subset \mathcal{H}(n)^G$ and $U \subset V_k(\mathfrak{g}, B)^G$ corresponding to S under the linear isomorphisms $\mathcal{H}(n)^G \cong R \cong V_k(\mathfrak{g}, B)^G$, are strong generating sets for $\mathcal{H}(n)^G$ and $V_k(\mathfrak{g}, B)^G$, respectively.

Even though S is a minimal generating set for R as a ∂ -ring, T and U need not be minimal strong generating sets for $\mathcal{H}(n)^G$ and $V_k(\mathfrak{g}, B)^G$, respectively, as vertex algebras. In fact, since $\mathcal{H}(n)^G$ is strongly finitely generated, there is a finite set

$$T' = \{p_1, \dots, p_s\},$$

which we may assume without loss of generality to be a subset of T , which strongly generates $\mathcal{H}(n)^G$. In other words, $\mathcal{H}(n)^G = \langle T' \rangle$, where $\langle T' \rangle$ denotes the space of normally ordered polynomials in p_1, \dots, p_s and their derivatives. Let $S' \subset S \subset R$ be the subset corresponding to T' under the linear isomorphism $\mathcal{H}(n)^G \cong R$, and let $\langle S' \rangle$ denote the space of polynomials in the elements of S' and their derivatives. In general, S' will not generate R as a ∂ -ring, so $\langle S' \rangle$ will be a proper subset of S . Since S is a minimal generating set for R as a ∂ -ring, it follows that

$$(6.1) \quad (S \setminus S') \cap \langle S' \rangle = \emptyset.$$

Otherwise such elements could be removed from S and the resulting set would still generate R as a ∂ -ring.

Let w_0 be the maximal weight of elements of T' . Without loss of generality, we may assume that T' contains all elements of T of weight at most w_0 . Let $q \in T$ be a vertex operator of weight $w > w_0$ and degree d , which must coincide with d_j for some $j = 1, \dots, r$, since these are the only possible degrees of elements of T . Since T' is a strong generating set for $\mathcal{H}(n)^G$, there exists a decoupling relation

$$(6.2) \quad q = P(p_1, \dots, p_s)$$

where P lies in $\langle T' \rangle$. Without loss of generality, let us choose P so that its leading term has minimal degree e . We may write

$$P = \sum_{a=d_1}^e P^a,$$

where d_1 is the minimal degree of the elements of T , and $P^a \in \langle T' \rangle$ is a homogeneous, normally ordered polynomial of total degree a . (In other words, P^a is a normal ordering of some homogeneous polynomial of degree a in the elements of S' and their derivatives). Moreover, since there are only double contractions in the vertex algebra $\mathcal{H}(n)$, we may assume that the degree of each nontrivial term appearing in P has the same parity as d . In other words, $P^a = 0$ whenever $a - d \equiv 1 \pmod{2}$. In particular, e has the same parity as d . It is clear from (6.1) that $e > d$ and that the leading term P^e is a normal ordering

of some relation in R ; as is Section 4, we may view (6.2) as a quantum correction of this relation. In fact, we can rewrite (6.2) in the form

$$(6.3) \quad \sum_{a=d_1}^e A^a = 0, \quad A^a = P^a \text{ for } a \neq d, \quad A^d = P^d - q.$$

It is convenient to introduce the parameter k into our Heisenberg algebra $\mathcal{H}(n)$. We denote by $\mathcal{H}_k(n)$ the vertex algebra with generators $\alpha^i(z)$, $i = 1, \dots, n$, and OPE relations

$$\alpha^i(z)\alpha^j(w) \sim k\delta_{i,j}(z-w)^{-2}.$$

Of course the vertex algebras $\mathcal{H}_k(n)$ are all isomorphic for $k \neq 0$, and for any reductive $G \subset O(n)$, the invariant subalgebras $\mathcal{H}_k(n)^G$ are isomorphic as well. The generators of $\mathcal{H}_k(n)^G$ are the same for all $k \neq 0$, but since each double contraction among the generators of $\mathcal{H}_k(n)$ introduces a factor of k , the decoupling relation corresponding to (6.2) becomes

$$(6.4) \quad k^{\frac{1}{2}(e-d)}q = \sum_{a=d_1}^e k^{\frac{1}{2}(e-a)}P^a.$$

We use the same notation $T' = \{p_1, \dots, p_s\}$ for our strong generating set, regarded now as a subset of $\mathcal{H}_k(n)^G$. As above, we may rewrite (6.4) in the form

$$(6.5) \quad \sum_{a=d_1}^e k^{\frac{1}{2}(e-a)}A^a = 0.$$

Let $U' \subset U \subset V_k(\mathfrak{g}, B)^G$ denote the subset corresponding to T' under the linear isomorphism $\mathcal{H}_k(n)^G \cong R \cong V_k(\mathfrak{g}, B)^G$. Let $\tilde{p}_i \in U'$ be the elements corresponding to $p_i \in T'$, and let $\langle U' \rangle$ denote the space of normally ordered polynomials in $\tilde{p}_1, \dots, \tilde{p}_s$ and their derivatives. Given $\tilde{q} \in U$ corresponding to $q \in T$, we will use the modified decoupling relation (6.4) to construct an analogous relation

$$(6.6) \quad \lambda(k)\tilde{q} = \sum_{a=d_1}^e Q^a,$$

for \tilde{q} in $V_k(\mathfrak{g}, B)^G$. Here Q^a is a normally ordered polynomial of total degree a in the elements of U and their derivatives, and the coefficient $\lambda(k)$ is a polynomial in k whose leading term is $k^{\frac{1}{2}(e-d)}$. Unlike P^a which lies in $\langle T' \rangle$, Q^a need not lie in $\langle U' \rangle$, so (6.6) need not be a decoupling relation, and will require further modification. If we work in an orthonormal basis $\{X^{\xi_i} \mid i = 1, \dots, n\}$ for \mathfrak{g} relative to \langle, \rangle , the second-order poles in the OPEs of $X^{\xi_i}(z)X^{\xi_j}(w)$ and $\alpha^i(z)\alpha^j(w)$ are the same, namely $k\delta_{i,j}(z-w)^{-2}$. Therefore passing from $\mathcal{H}_k(n)$ to $V_k(\mathfrak{g}, B)$ amounts to “turning on” the first-order polar part of the OPEs among the generators X^{ξ_i} of $V_k(\mathfrak{g}, B)$.

Given a normally ordered polynomial $P \in \mathcal{H}_k(n)^G$ in the elements of T , let $\tilde{P} \in V_k(\mathfrak{g}, B)^G$ denote the corresponding normally ordered polynomial in the elements of U . Since $gr(\mathcal{H}_k(n))^G \cong R \cong gr(V_k(\mathfrak{g}, B))^G$ as graded rings, \tilde{P}^e is a normal ordering of a relation in R . As in Section 4, there exist $B^a \in V_k(\mathfrak{g}, B)^G$ of degree a for $a = d_1, \dots, e$ with $B^e = \tilde{P}^e$, and $\sum_{a=d_1}^e B^a = 0$. Note that the indices a need not be parity-homogeneous because there are both single and double contractions in $V_k(\mathfrak{g}, B)^G$. There is a lot of flexibility in the choice of these quantum corrections, and as we shall see, certain pieces have to be chosen with some care.

The important observation is that the coefficients of all monomials appearing in B^a are polynomials in k . The leading degree of such a coefficient, regarded as a polynomial in k , will be called the k -degree. It is clear from the OPE relations in $V_k(\mathfrak{g}, B)$ that each double contraction among the generators X^{ξ_i} , which lowers the total degree by two, will introduce a factor of k , and each single contraction, which lowers the degree by one, will not introduce a factor of k . For each term B^a for which a has the same parity as e , the highest k -degree that can occur is $\frac{1}{2}(e - a)$, which occurs only if every possible contraction was a double contraction. These terms are therefore the same as the corresponding terms in the relation (6.5) in $\mathcal{H}_k(n)^G$. More precisely, for all a of the same parity as e , we may choose B^a so that

$$(6.7) \quad B^a = k^{\frac{1}{2}(e-a)} \tilde{A}^a + \dots,$$

where (\dots) consists of monomials whose coefficients have lower k -degree. Note also that when a has the same parity as e , the coefficient of each term appearing in B^{a-1} has k -degree at most $\frac{1}{2}(e - a)$.

In degree d , since $A^d = P^d - q$ and $P^d \in \langle T' \rangle$, it follows that

$$(6.8) \quad B^d = k^{\frac{1}{2}(e-d)} \tilde{P}^d - k^{\frac{1}{2}(e-d)} \tilde{q} + (\dots),$$

where the coefficient of each term appearing in (\dots) has k -degree at most $\frac{1}{2}(e - d) - 1$. If \tilde{q} appears in (\dots) , we can rewrite (6.8) in the form

$$B^d = k^{\frac{1}{2}(e-d)} \tilde{P}^d - \lambda(k) \tilde{q} + (\dots),$$

where \tilde{q} does not appear in (\dots) , and $\lambda(k)$ is a polynomial in k whose leading term is $k^{\frac{1}{2}(e-d)}$. Finally, letting $Q^a = B^a$ for $a \neq d$ and $Q^d = B^d + \lambda(k) \tilde{q}$, we obtain a relation $\lambda(k) \tilde{q} = \sum_{a=d_1}^e Q^a$ of the form (6.6). Note that Q^d is independent of \tilde{q} , and whenever a and e have the same parity, Q^a satisfies

$$Q^a = k^{\frac{1}{2}(e-a)} \tilde{P}^a + \dots,$$

where the coefficient of each term in (\dots) has lower k -degree. For $a < e$, the term \tilde{P}^a lies in $\langle U' \rangle$, but the other terms appearing in Q^a need not lie in $\langle U' \rangle$, and can depend on other elements of U . We need to systematically eliminate these variables for generic values of k , and construct a *decoupling* relation expressing \tilde{q} as an element of $\langle U' \rangle$.

We begin with the elements of weight $w_0 + 1$. Recall that d_1, \dots, d_r are the distinct degrees of the elements of U , with $d_1 < d_2 < \dots < d_r$. For $j = 1, \dots, r$, let $\{\tilde{q}_{j,1}, \dots, \tilde{q}_{j,t_j}\}$ be the set of elements of U of weight $w_0 + 1$ and degree d_j . Let

$$\{q_{j,1}, \dots, q_{j,t_j}\} \subset T \subset \mathcal{H}_k(n)^G, \quad \{s_{j,1}, \dots, s_{j,t_j}\} \subset S \subset R$$

be the sets corresponding to $\{\tilde{q}_{j,1}, \dots, \tilde{q}_{j,t_j}\}$ under the linear isomorphisms $V_k(\mathfrak{g}, B)^G \cong R \cong \mathcal{H}_k(n)^G$. First we consider the elements $\tilde{q}_{1,1}, \dots, \tilde{q}_{1,t_1}$ of minimal degree d_1 . For each of the corresponding elements $q_{1,i} \in T$, there is a decoupling relation

$$(6.9) \quad k^{\frac{1}{2}(e_1-i-d_1)} q_{1,i} = \sum_{a=d_1}^{e_{1,i}} k^{\frac{1}{2}(e_{1,i}-a)} P_{1,i}^a$$

in $\mathcal{H}_k(n)^G$ of the form (6.4), where $e_{1,i}$ has the same parity as d_1 , and each $P_{1,i}^a$ lies in $\langle T' \rangle$. There is a corresponding relation

$$(6.10) \quad \lambda_{1,i}(k) \tilde{q}_{1,i} = \sum_{a=d_1}^{e_{1,i}} Q_{1,i}^a$$

in $V_k(\mathfrak{g}, B)^G$ of the form (6.6), where the coefficient $\lambda_{1,i}(k)$ has k -degree $\frac{1}{2}(e_{1,i} - d_1)$, and $\tilde{q}_{1,i}$ does not appear in $Q_{1,i}^{d_1}$. Since the term $P_{1,i}^{d_1}$ appearing in (6.9) lies in $\langle T' \rangle$, and $q_{1,j}$ lies in $T \setminus T'$ for $j = 1, \dots, t_1$, it follows from (6.1) that $q_{1,j}$ does not appear in $P_{1,i}^{d_1}$. Therefore the coefficient of $\tilde{q}_{1,j}$ in $Q_{1,i}^{d_1}$ for $j \neq i$ can have k -degree at most $\frac{1}{2}(e_{1,i} - d_1) - 1$. We may write $Q_{1,i}^{d_1}$ in the form

$$Q_{1,i}^{d_1} = \sum_{j=1}^{t_1} \epsilon_{i,j}(k) \tilde{q}_{1,j} + \dots,$$

where $\epsilon_{i,i}(k) = 0$, and the term (\dots) lies in $\langle U' \rangle$. Clearly each $\epsilon_{i,j}(k)$ for $j \neq i$ has k -degree at most $\frac{1}{2}(e_{1,i} - d_1) - 1$. We can rewrite these relations in the form of a linear system of t_1 equations

$$(6.11) \quad \sum_{j=1}^{t_1} \mu_{i,j}(k) \tilde{q}_{1,j} = R_{1,i}, \quad i = 1, \dots, t_1.$$

In this notation, $\mu_{i,j}(k) = \epsilon_{i,j}(k)$ for $j \neq i$, and $\mu_{i,i}(k) = -\lambda_{1,i}(k)$. Moreover, the vertex operator $R_{1,i} = \sum_{a=d_1}^{e_{1,i}} R_{1,i}^a$ has the property that $R_{1,i}^{d_1} \in \langle U' \rangle$, since all the elements of degree d_1 that do *not* lie in $\langle U' \rangle$ now appear on the left hand side of (6.11). In fact, we claim that for $d_1 < a < d_2$, each term $R_{1,i}^a$ lies in $\langle U' \rangle$. Since every element of U has degree d_i for some $i = 1, \dots, r$, and $d_1 < a < d_2$, it follows that $R_{1,i}^a$ is a normally ordered polynomial in the degree d_1 elements of U and their derivatives. Since $a > d_1$, each term in $R_{1,i}^a$ must be *nonlinear* in these generators, and in particular a must be a multiple of d_1 . Since the weight of $R_{1,i}^a$ is $w_0 + 1$, each factor of degree d_1 appearing in each monomial can have weight at most w_0 , and hence will lie in $\langle U' \rangle$.

Since the diagonal entries $\mu_{i,i}(k)$ appearing in (6.11) have k -degree $\frac{1}{2}(e_{1,i} - d_1)$, and all other entries $\mu_{i,j}(k)$ for $j \neq i$ have lower k -degree, the determinant of this system is a nonzero polynomial of leading k -degree $\sum_{i=1}^{t_1} \frac{1}{2}(e_{1,i} - d_1)$. Hence this system is invertible over the fraction field $\mathbb{C}(k)$, so we obtain new relations of the form

$$(6.12) \quad \tilde{q}_{1,i} = \sum_{a=d_1}^{e'_{1,i}} S_{1,i}^a, \quad i = 1, \dots, t_1.$$

Here $S_{1,i}^a$ is a normally ordered polynomial in the elements of U and their derivatives, which is homogeneous of degree a . By construction, $S_{1,i}^a$ lies in $\langle U' \rangle$ for $d_1 \leq a < d_2$. Note that the indices $e'_{1,i}$ need not be the same as the indices $e_{1,i}$ appearing in the old relations (6.10).

Clearly the higher terms $S_{1,i}^a$ for $a \geq d_2$ need not lie in $\langle U' \rangle$, so we need to iterate this procedure and further modify the relations (6.12). Recall that $\tilde{q}_{2,1}, \dots, \tilde{q}_{2,t_2} \in U$ are the elements of degree d_2 and weight $w_0 + 1$, and $q_{2,1}, \dots, q_{2,t_2}$ are the corresponding elements

of $T \subset \mathcal{H}_k(n)^G$. There are decoupling relations in $\mathcal{H}_k(n)^G$

$$(6.13) \quad k^{\frac{1}{2}(e_{2,i}-d_2)} q_{2,i} = \sum_{a=d_1}^{e_{2,i}} k^{\frac{1}{2}(e_{2,i}-a)} P_{2,i}^a$$

of the form (6.4), where $e_{2,i}$ has the same parity as d_2 , and $P_{2,i}^a$ is a homogeneous normally ordered polynomial of degree a in $\langle T' \rangle$. The corresponding relation in $V_k(\mathfrak{g}, B)^G$ is of the form

$$(6.14) \quad \lambda_{2,i}(k) \tilde{q}_{2,i} = \sum_{a=d_1}^{e_{2,i}} Q_{2,i}^a$$

where $\lambda_{2,i}(k)$ has k -degree $\frac{1}{2}(e_{2,i} - d_2)$. Moreover, the leading term $Q_{2,i}^{e_{2,i}} = \tilde{P}_{2,i}^{e_{2,i}}$, and for each a of the same parity as d_2 , we have $Q_{2,i}^a = k^{\frac{1}{2}(e_{2,i}-a)} \tilde{P}_{2,i}^a + \dots$, where the coefficient of each monomial appearing in (\dots) has k -degree at most $\frac{1}{2}(e_{2,i} - a) - 1$. In degree d_2 , $Q_{2,i}^{d_2}$ is independent of $\tilde{q}_{2,i}$, but may depend on the elements $\tilde{q}_{2,j}$ for $j \neq i$. However, the coefficient of $\tilde{q}_{2,j}$ in $Q_{2,i}^{d_2}$ will have k -degree at most $\frac{1}{2}(e_{2,i} - d_2) - 1$.

By degree and weight considerations, the terms $Q_{2,i}^a$ for $d_1 < a < d_2$ each lie in $\langle U' \rangle$. The argument is the same as the proof of the corresponding statement for $R_{1,i}^a$. On the other hand, the degree d_1 term $Q_{2,i}^{d_1}$ in (6.14) need not lie in $\langle U' \rangle$, and may depend on the elements $\tilde{q}_{1,1}, \dots, \tilde{q}_{1,t_1}$. We need to eliminate these terms using the relations (6.12) constructed above. Suppose first that d_1 and d_2 have the same parity. Since the degree d_1 term $P_{2,i}^{d_1} \in \mathcal{H}_k(n)^G$ in (6.13) lies in $\langle T' \rangle$, it follows that the coefficient of each term $\tilde{q}_{1,1}, \dots, \tilde{q}_{1,t_1}$ appearing in $Q_{2,i}^{d_1}$ can have k -degree at most $\frac{1}{2}(e_{2,i} - d_1) - 1$. If $\tilde{q}_{1,l}$ appears in $Q_{2,i}^{d_1}$ for some $l = 1, \dots, t_1$, we can use (6.12) in the case $i = l$ to eliminate it. Of course this can introduce a new multiple of $\tilde{q}_{2,j}$ for $j = 1, \dots, t_2$, in degree d_2 . However, the maximal k -degree of terms appearing in $S_{1,l}^{d_2}$ is $-\frac{1}{2}(d_2 - d_1)$, which is a negative integer since (6.12) has been normalized so that the k -degree of the left hand side is zero. Hence the coefficients of these new multiples of $\tilde{q}_{2,j}$ must be rational functions of k , of k -degree at most

$$\frac{1}{2}(e_{2,i} - d_1) - 1 - \frac{1}{2}(d_2 - d_1) = \frac{1}{2}(e_{2,i} - d_2) - 1.$$

Since the coefficient $\lambda_{2,i}(k)$ of $\tilde{q}_{2,i}$ in (6.14) has k -degree $\frac{1}{2}(e_{2,i} - d_2)$, any multiple of $\tilde{q}_{2,i}$ introduced by our procedure of eliminating the degree d_1 terms $\tilde{q}_{1,l}$ cannot cancel the term $\lambda_{2,i}(k) \tilde{q}_{2,i}$. So we can collect terms involving $\tilde{q}_{2,i}$ and rewrite our relation in the form

$$(6.15) \quad \nu_{2,i}(k) \tilde{q}_{2,i} = \sum_{a=d_1}^{e'_{2,i}} R_{2,i}^a,$$

where the coefficient $\nu_{2,i}(k)$ is a rational function of k -degree $\frac{1}{2}(e_{2,i} - d_2)$, and $R_{2,i}^a$ lies in $\langle U' \rangle$ for $d_1 \leq a < d_2$. Note that our elimination procedure may have introduced terms in degree greater than $e_{2,i}$, so it is possible that $e'_{2,i} > e_{2,i}$. Finally, the degree d_2 term $R_{2,i}^{d_2}$ is independent of $\tilde{q}_{2,i}$, and still has the property that the coefficient of any term of the form $\tilde{q}_{2,j}$ for $j \neq i$ has k -degree at most $\frac{1}{2}(e_{2,i} - d_2) - 1$.

Next, suppose that d_1 and d_2 have opposite parity. Since the parity of $e_{2,i}$ is the same as the parity of d_2 (and hence opposite to the parity of d_1), the coefficient of each term in $Q_{2,i}^{d_1}$

can have k -degree at most $\frac{1}{2}(e_{2,i} - d_1 - 1)$. Any terms appearing in $Q_{2,i}^{d_1}$ of the form $\tilde{q}_{1,l}$ can be eliminated using the relations (6.12). Since the maximal k -degree of terms appearing in $S_{1,l}^{d_2}$ is $-\frac{1}{2}(d_2 - d_1 + 1)$, this procedure may introduce new terms of the form $\tilde{q}_{2,j}$ in degree d_2 , but their coefficients will be rational functions in k , of k -degree at most

$$\frac{1}{2}(e_{2,i} - d_1 - 1) - \frac{1}{2}(d_2 - d_1 + 1) = \frac{1}{2}(e_{2,i} - d_2) - 1.$$

Thus we can collect terms involving $\tilde{q}_{2,i}$ as above, and we obtain a similar relation of the form (6.15), where $R_{2,i}^a$ lies in $\langle U' \rangle$ for $d_1 \leq a < d_2$. As above, $R_{2,i}^{d_2}$ is independent of $\tilde{q}_{2,i}$, but may depend on $\tilde{q}_{2,j}$ for $j \neq i$. However, the coefficient of such terms can have k -degree at most $\frac{1}{2}(e_{2,i} - d_2) - 1$.

Regardless of whether d_1 and d_2 have the same or opposite parity, each monomial appearing in $R_{2,i}^a$ for $d_2 < a < d_3$ must be nonlinear in the elements of U of degrees d_1 or d_2 , and their derivatives. Since the weight of each factor can be at most w_0 , it follows that $R_{2,i}^a \in \langle U' \rangle$ for $d_2 < a < d_3$. The degree d_2 term $R_{2,i}^{d_2}$ need not lie in $\langle U' \rangle$, but we can eliminate the terms $\tilde{q}_{2,j}$ for $j \neq i$ which do not lie in $\langle U' \rangle$ using the same procedure we used in the degree d_1 case. First, we rewrite the relations (6.15) in the form of a linear system of t_2 equations

$$\sum_{j=1}^{t_2} \mu_{i,j}(k) \tilde{q}_{2,j} = S_{2,i}, \quad i = 1, \dots, t_2.$$

Here $\mu_{i,i}(k) = -\nu_{2,i}(k)$, which has k -degree $\frac{1}{2}(e_{2,i} - d_2)$, and $\mu_{i,j}(k)$ has k -degree at most $\frac{1}{2}(e_{2,i} - d_2) - 1$ for $j \neq i$. Moreover, the vertex operators $S_{2,i} = \sum_{a=d_1}^{e'_{2,i}} S_{2,i}^a$ have the property that $S_{2,i}^a \in \langle U' \rangle$ for $d_1 \leq a < d_3$. As in the case of (6.11), the corresponding determinant is invertible over $\mathbb{C}(k)$, so we obtain new relations

$$\tilde{q}_{2,i} = \sum_{a=d_1}^{e''_{2,i}} T_{2,i}^a$$

for $i = 1, \dots, t_2$, where by construction, $T_{2,i}^a$ lies in $\langle U' \rangle$ for $d_1 \leq a < d_3$. Note that the upper indices $e''_{2,i}$ may be different from $e'_{2,i}$. Of course $T_{2,i}^a$ need not lie in $\langle U' \rangle$ for $a \geq d_3$.

Now, using the same procedure, for each element $\tilde{q}_{3,1}, \dots, \tilde{q}_{3,t_3} \in U$ of weight $w_0 + 1$ and degree d_3 , we can construct a relation of the form

$$\tilde{q}_{3,i} = \sum_{a=d_1}^{e_{3,i}} T_{3,i}^a,$$

where $T_{3,i}^a \in \langle U' \rangle$ for $d_1 \leq a < d_4$. Continuing this process, in degree d_{j-1} , each element $\tilde{q}_{j-1,i}$ for $i = 1, \dots, t_{j-1}$ satisfies a relation

$$(6.16) \quad \tilde{q}_{j-1,i} = \sum_{a=d_1}^{e_{j-1,i}} T_{j-1,i}^a,$$

where $T_{j-1,i}^a \in \langle U' \rangle$ for $d_1 \leq a < d_j$. Finally, in the highest degree d_r among elements of U , we have honest decoupling relations

$$\tilde{q}_{r,i} = \sum_{a=d_1}^{e_{r,i}} T_{r,i}^a$$

for $i = 1, \dots, t_r$, where each $T_{r,i}^a$ lies in $\langle U' \rangle$. Now we can substitute these relations into the relations (6.16) for $\tilde{q}_{r-1,i}$ in the case $j = r$, eliminating all variables of the form $\tilde{q}_{r,i}$ that occur in $T_{r-1,i}^a$ for $a \geq d_r$. This produces decoupling relations expressing each $\tilde{q}_{r-1,i}$ as an element of $\langle U' \rangle$, for $i = 1, \dots, t_{r-1}$.

Inductively, once we have constructed decoupling relation for all elements of the form $\tilde{q}_{k,l}$ with $j \leq k \leq r$ and $l = 1, \dots, t_k$, we can eliminate all variables of the form $\tilde{q}_{k,l}$ for $j \leq k \leq r$ in the relations (6.16) for $\tilde{q}_{j-1,i}$. We obtain decoupling relations expressing each $\tilde{q}_{j-1,i}$ for $i = 1, \dots, t_{j-1}$, as an element of $\langle U' \rangle$. At the end of this procedure, we have a decoupling relation for each element of U of weight $w_0 + 1$. We repeat this entire procedure for weights $w_0 + 2$, $w_0 + 3$, etc. This shows that if we work over the field $\mathbb{C}(k)$, we can find a complete set of decoupling relations, expressing each element of U as an element of $\langle U' \rangle$. Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. For arbitrary k , let $\mathcal{W}(\mathfrak{g}, B, G)_k \subset V_k(\mathfrak{g}, B)^G$ denote the vertex subalgebra generated by U' . Clearly $\langle U' \rangle \subset \mathcal{W}(\mathfrak{g}, B, G)_k$, and we have $\langle U' \rangle = \mathcal{W}(\mathfrak{g}, B, G)_k$ precisely when $\mathcal{W}(\mathfrak{g}, B, G)_k$ is *strongly* generated by U' . In each weight w , the construction of our decoupling relations required us to invert a finite set of determinants over $\mathbb{C}(k)$ which are certain rational function of k , so finitely many values of k must be excluded. Taking the union of these excluded values over all weights, it follows that the set of k for which the equality

$$(6.17) \quad \langle U' \rangle = \mathcal{W}(\mathfrak{g}, B, G)_k = V_k(\mathfrak{g}, B)^G$$

can fail is at most countable. Therefore (6.17) holds for generic values of k . \square

Recall that a *deformable \mathcal{W} -algebra* is a family of vertex algebras \mathcal{W}_k , equipped with strong generating sets $A_k = \{a_1^k, \dots, a_r^k\}$, whose structure constants are continuous functions of k with isolated singularities. The structure constants are just the coefficients of each normally ordered monomial in the elements of A_k and their derivatives appearing in the OPE of $a_i^k(z)a_j^k(w)$, for $i, j = 1, \dots, r$.

Proof of Corollary 1.2. We claim that the vertex algebra $\mathcal{W}(\mathfrak{g}, B, G)_k$ generated by U' as above is a deformable \mathcal{W} -algebra. Let $E \subset \mathbb{C}$ be the set of all values of k for which $\langle U' \rangle = \mathcal{W}(\mathfrak{g}, B, G)_k$. A priori, it is possible that the set D on which (6.17) holds, is a proper subset of E . In other words, there may be values of k for which $\langle U' \rangle = \mathcal{W}(\mathfrak{g}, B, G)_k$, but $\mathcal{W}(\mathfrak{g}, B, G)_k$ is a proper subset of $V_k(\mathfrak{g}, B)^G$. For $k \in E$, the structure constants of $\mathcal{W}(\mathfrak{g}, B, G)_k$ are clearly rational functions of k . Since there are only finitely many structure constants, there can be only finitely many singular points k_1, \dots, k_t , and we have $E = \mathbb{C} \setminus \{k_1, \dots, k_t\}$. \square

Unlike the algebras $\mathcal{W}(\mathfrak{g}, f)_k$ associated via quantum Drinfeld-Sokolov reduction to a simple, finite-dimensional Lie algebra \mathfrak{g} and a nilpotent element $f \in \mathfrak{g}$, the vertex algebras $\mathcal{W}(\mathfrak{g}, B, G)_k$ are in general *not* freely generated. To see this, let \mathcal{V} be a vertex algebra in the category \mathcal{R} , equipped with a good increasing filtration $\mathcal{V}_{(0)} \subset \mathcal{V}_{(1)} \subset \dots$. Suppose that \mathcal{V} is freely generated by some collection $\{a_i(z) | i \in I\}$, where $a_i(z) \in \mathcal{V}_{(d_i)} \setminus \mathcal{V}_{(d_i-1)}$. Since there are no normally ordered polynomial relations among the generators and their derivatives, it follows that $gr(\mathcal{V})$ is the polynomial algebra with generators $a_i = \phi_{d_i}(a_i(z))$ and their derivatives. Hence \mathcal{V} cannot be freely generated if $gr(\mathcal{V})$ is not a polynomial

algebra. In general, for a group G and linear G -representation V , $(\text{Sym} \bigoplus_{j \geq 0} V_j)^G$ is not a polynomial algebra, so $\mathcal{W}(\mathfrak{g}, B, G)_k$ will not be freely generated.

7. A CONCRETE EXAMPLE: $\mathfrak{g} = \mathfrak{sl}_2$ AND $G = SL(2)$

In this section we consider the $SL(2)$ -invariant subalgebra of $V_k(\mathfrak{sl}_2)$ in order to give the reader a feeling for the structure of the invariant vertex algebras studied in this paper. The vertex algebra $V_k(\mathfrak{sl}_2)^{SL(2)}$ was previously studied in [BFH], where it was conjectured to be strongly finitely generated. We work in the usual root basis x, y, h satisfying

$$[x, y] = h, \quad [h, x] = 2x, \quad [h, y] = -2y.$$

The generators $X^x(z), X^y(z), X^h(z)$ of $V_k(\mathfrak{sl}_2)$ satisfy

$$\begin{aligned} X^x(z)X^y(w) &\sim k(z-w)^{-2} + X^h(z-w)^{-1}, & X^h(z)X^h(w) &\sim 2k(z-w)^{-2}, \\ X^h(z)X^x(w) &\sim 2X^x(z-w)^{-1}, & X^h(z)X^y(w) &\sim -2X^y(z-w)^{-1}. \end{aligned}$$

We need the following theorem of Weyl [We] which describes polynomial invariants for the adjoint representation of \mathfrak{sl}_2 .

Theorem 7.1. *For $n \geq 0$, let V_n be a copy of the adjoint representation of \mathfrak{sl}_2 , with basis $\{a_n^h, a_n^x, a_n^y\}$. Then $(\text{Sym} \bigoplus_{n=0}^{\infty} V_n)^{\mathfrak{sl}_2}$ is generated by*

$$(7.1) \quad q_{ij} = a_i^h a_j^h + 2a_i^x a_j^y + 2a_i^y a_j^x, \quad i, j \geq 0,$$

$$(7.2) \quad c_{klm} = \begin{vmatrix} a_k^h & a_k^x & a_k^y \\ a_l^h & a_l^x & a_l^y \\ a_m^h & a_m^x & a_m^y \end{vmatrix}, \quad 0 \leq k < l < m.$$

The ideal of relations among the variables q_{ij} and c_{klm} is generated by polynomials of the following two types:

$$(7.3) \quad q_{ij}c_{klm} - q_{kj}c_{ilm} + q_{lj}c_{kim} - q_{mj}c_{kli},$$

$$(7.4) \quad c_{ijk}c_{lmn} + \frac{1}{4} \begin{vmatrix} q_{il} & q_{im} & q_{in} \\ q_{jl} & q_{jm} & q_{jn} \\ q_{kl} & q_{km} & q_{kn} \end{vmatrix}.$$

Recall that $\mathfrak{sl}_2 \cong \mathfrak{so}_3$ as complex Lie algebras, and the adjoint representation V of \mathfrak{sl}_2 is isomorphic to the standard representation W of \mathfrak{so}_3 . Hence for $V_j \cong V$ and $W_j \cong W$, the invariant rings $(\text{Sym} \bigoplus_{j \geq 0} V_j)^{\mathfrak{sl}_2}$ and $(\text{Sym} \bigoplus_{j \geq 0} W_j)^{\mathfrak{so}_3}$ are isomorphic, and we may identify these rings. Recall that $SO(3)$ acts on the Heisenberg algebra $\mathcal{H}(3)$, and we have linear isomorphisms

$$(7.5) \quad \mathcal{H}(3)^{SO(3)} \cong \text{gr}(\mathcal{H}(3))^{SO(3)} \cong (\text{Sym} \bigoplus_{j \geq 0} V_j)^{\mathfrak{sl}_2} \cong \text{gr}(V_k(\mathfrak{sl}_2))^{SL(2)} \cong V_k(\mathfrak{sl}_2)^{SL(2)}$$

and isomorphisms of ∂ -rings

$$(7.6) \quad \text{gr}(\mathcal{H}(3))^{SO(3)} \cong (\text{Sym} \bigoplus_{j \geq 0} V_j)^{\mathfrak{sl}_2} \cong \text{gr}(V_k(\mathfrak{sl}_2))^{SL(2)}.$$

Under (7.5), the generating set $\{q_{ij}, c_{klm}\}$ for $(\text{Sym} \bigoplus_{j \geq 0} V_j)^{\mathfrak{sl}_2}$ corresponds to strong generating sets $\{Q_{ij}, C_{klm}\}$ for $\mathcal{H}(3)^{SO(3)}$ and $\{\tilde{Q}_{ij}, \tilde{C}_{klm}\}$ for $V_k(\mathfrak{sl}_2)^{SL(2)}$, respectively, by Lemma

3.1. In this notation, Q_{ij} corresponds to the element $\omega_{i,j}$ given by (5.5), and $Q_{0,2m}$ corresponds to j^{2m} , which is given by (5.6). Note that Q_{ij} and \tilde{Q}_{ij} have weight $i + j + 2$, and C_{klm} and \tilde{C}_{klm} have weight $k + l + m + 3$. In terms of the generators X^x, X^y, X^h for $V_k(\mathfrak{sl}_2)$, we have

$$(7.7) \quad \tilde{Q}_{ij} =: \partial^i X^h \partial^j X^h : + 2 : \partial^i X^x \partial^j X^y : + 2 : \partial^i X^y \partial^j X^x :,$$

$$(7.8) \quad \begin{aligned} \tilde{C}_{klm} =: & \partial^k X^x \partial^l X^y \partial^m X^h : - : \partial^k X^x \partial^m X^y \partial^l X^h : - : \partial^l X^x \partial^k X^y \partial^m X^h : + : \partial^l X^x \partial^m X^y \partial^k X^h : \\ & + : \partial^m X^x \partial^k X^y \partial^l X^h : - : \partial^m X^x \partial^l X^y \partial^k X^h : . \end{aligned}$$

It is clear from Theorem 7.1 that $(Sym \bigoplus_{j \geq 0} V_j)^{\mathfrak{sl}_2}$ is not a polynomial ring, so $V_k(\mathfrak{sl}_2)^{SL(2)}$ is not freely generated by any set of vertex operators.

Next, we find a strong finite generating set for $\mathcal{H}(3)^{SO(3)}$, using the structure of $\mathcal{H}(3)^{SO(3)}$ as a module over $\mathcal{H}(3)^{O(3)}$ and the fact that $\mathcal{H}(3)^{O(3)}$ is a \mathcal{W} -algebra of type $\mathcal{W}(2, 4, 6, \dots, 18)$. By Theorem 6.1 of [LIII], for any $G \subset O(3)$, there is a finite set of irreducible, highest-weight $\mathcal{H}(3)^{O(3)}$ -submodules of $\mathcal{H}(3)$ whose direct sum contains a strong generating set for $\mathcal{H}(3)^G$. In the case $G = SO(3)$, we only need two such modules \mathcal{M}_0 and \mathcal{M} , where $\mathcal{M}_0 = \mathcal{H}(3)^{O(3)}$, which has highest-weight vector 1 and contains all the quadratics Q_{ij} , and \mathcal{M} has highest-weight vector C_{012} and contains all the cubics C_{klm} .

Recall from [LIII] that the Lie algebra \mathcal{P} generated by $\{j^{2l}(k) \mid k, l \geq 0\}$ acts by derivations of degree zero on $gr(\mathcal{H}(3))^{SO(n)}$. Let $\mathcal{M}' \subset \mathcal{M}$ be the \mathcal{P} -submodule of \mathcal{M} generated by C_{012} , which lies in the filtered component $\mathcal{M}_{(3)}$ since $C_{012} \in \mathcal{M}_{(3)}$ and \mathcal{P} preserves the filtration. By Lemma 6.6 of [LIII], \mathcal{M}' is spanned by elements of the form

$$(7.9) \quad j^{2l_1}(k_1)j^{2l_2}(k_2)j^{2l_3}(k_3)C_{012}, \quad k_i \leq 5,$$

since we have $d = 3$ and $m = 2$ in this case. Without loss of generality, we may assume that $l_i \geq l_j$ for $i, j = 1, 2, 3$, and $k_i \leq k_j$ whenever $l_i = l_j$. Letting $\mathcal{H}(3)^{O(3)}[k]$ denote the subspace of conformal weight k , define the *Wick ideal* $\mathcal{M}_{Wick} \subset \mathcal{M}$ to be the subspace spanned by

$$: a(z)b(z) :, \quad a(z) \in \bigoplus_{k > 0} \mathcal{H}(3)^{O(3)}[k], \quad b(z) \in \mathcal{M}.$$

By Lemma 6.7 of [LIII], $\mathcal{M}/\mathcal{M}_{Wick}$ is finite-dimensional, so there exists a finite set $S \subset \mathcal{M}$ such that \mathcal{M} is spanned by the elements

$$: \omega_1(z) \cdots \omega_t(z) \alpha(z) :, \quad \omega_j(z) \in \mathcal{H}(3)^{O(3)}, \quad \alpha(z) \in S.$$

It follows from the proof of Lemma 6.7 of [LIII] that S can be taken to be a subset of \mathcal{M}' . We claim that S can be taken to be any basis for the subspace of \mathcal{M}' of weight at most 327. Since $j^{2l}(k)$ has weight $2l - k + 1$ and C_{012} has weight 6, $(j^{106}(0))^3 C_{012}$ has weight 327. It follows that for any element

$$\omega = j^{2l_1}(k_1)j^{2l_2}(k_2)j^{2l_3}(k_3)C_{012} \in \mathcal{M}'$$

of the form (7.9) of weight greater than 327, we must have $2l_1 \geq 108$. The minimal weight of $j^{2l_1}(k_1)$ is 104 and occurs if $2l_1 = 108$ and $k_1 = 5$. Using the decoupling relation $j^{2l_1} = Q_{2l_1}(j^0, \dots, j^{16})$, the operator $j^{2l_1}(k_1)$ can be expressed as a linear combination of operators of the form

$$(7.10) \quad j^{2r_1}(s_1) \cdots j^{2r_t}(s_t), \quad 2r_i \leq 16, \quad s_i \in \mathbb{Z}.$$

Using the formula

$$(7.11) \quad (: ab :) \circ_n c = \sum_{k \geq 0} \frac{1}{k!} : (\partial^k a)(b \circ_{n+k} c) : + (-1)^{|a||b|} \sum_{k \geq 0} b \circ_{n-k-1} (a \circ_k c),$$

which holds for any $n > 0$ and any vertex operators a, b, c in a vertex algebra \mathcal{A} , it is easy to check that $j^{2l_1}(k_1)$ is a linear combination of terms of the form (7.10) with $1 \leq t \leq 6$, and $\sum_{i=1}^t s_i = 6 - t$. The maximal weight of such terms involving no creation operators is 102, since $(j^{16}(0))^6$ has weight 102. Since $j^{2l_1}(k_1)$ has weight at least 104, each term of the form (7.10) will involve creation operators. Therefore ω lies in the Wick ideal.

Since all terms of the form (7.9) are linear combinations of the C_{klm} , we may take

$$S = \{C_{klm} \mid wt(C_{klm}) = k + l + m + 3 \leq 327\}.$$

Theorem 6.9 of [LIII] then shows that $S \cup \{Q_{0,2j} \mid 0 \leq j \leq 8\}$ is a strong finite generating set for $\mathcal{H}(3)^{SO(3)}$. Finally, by combining this with our main result we obtain

Theorem 7.2. *The corresponding set*

$$\{\tilde{C}_{klm} \mid k + l + m \leq 324\} \cup \{\tilde{Q}_{0,2j} \mid 0 \leq j \leq 8\} \subset V_k(\mathfrak{sl}_2)^{SL(2)},$$

where \tilde{Q}_{ij} and \tilde{C}_{klm} are given by (7.7)-(7.8), is a strong finite generating set for $V_k(\mathfrak{sl}_2)^{SL(2)}$ for generic values of k .

Computer calculations indicate that this strong generating set is much larger than necessary. We hope to return to the question of finding minimal strong generating sets for these vertex algebras in the future.

REFERENCES

- [BFH] J. de Boer, L. Feher, and A. Honecker, *A class of \mathcal{W} -algebras with infinitely generated classical limit*, Nucl. Phys. B420 (1994), 409-445.
- [B] R. Borcherds, *Vertex operator algebras, Kac-Moody algebras and the monster*, Proc. Nat. Acad. Sci. USA 83 (1986) 3068-3071.
- [DSK] A. de Sole and V. Kac, *Finite vs affine \mathcal{W} -algebras*, Jpn. J. Math. 1 (2006), no. 1, 137-261.
- [DN] C. Dong and K. Nagatomo, *Classification of irreducible modules for the vertex operator algebra $M(1)^+$* , J. Algebra 216 (1999) no. 1, 384-404.
- [EFH] W. Eholzer, L. Feher, and A. Honecker, *Ghost systems: a vertex algebra point of view*, Nuclear Phys. B 518 (1998), no. 3, 669-688.
- [FKRW] E. Frenkel, V. Kac, A. Radul, and W. Wang, *$\mathcal{W}_{1+\infty}$ and $\mathcal{W}(\mathfrak{gl}_N)$ with central charge N* , Commun. Math. Phys. 170 (1995), 337-358.
- [FBZ] E. Frenkel and D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*, Math. Surveys and Monographs, Vol. 88, American Math. Soc., 2001.
- [FLM] I.B. Frenkel, J. Lepowsky, and A. Meurman, *Vertex Operator Algebras and the Monster*, Academic Press, New York, 1988.
- [K] V. Kac, *Vertex Algebras for Beginners*, University Lecture Series, Vol. 10. American Math. Soc., 1998.
- [KR] V. Kac and A. Radul, *Representation theory of the vertex algebra $\mathcal{W}_{1+\infty}$* , Transformation Groups, Vol. 1 (1996) 41-70.
- [KRW] V. Kac, S. Roan, and M. Wakimoto, *Quantum reduction for affine superalgebras*, Comm. Math. Phys. 241 (2003), no. 2-3, 307-342.
- [KWY] V. Kac, W. Wang, and C. Yan, *Quasifinite representations of classical Lie subalgebras of $\mathcal{W}_{1+\infty}$* , Advances in Mathematics, vol. 139 (1), (1998) 59-140.
- [Li] H. Li, *Vertex algebras and vertex Poisson algebras*, Commun. Contemp. Math. 6 (2004) 61-110.
- [LL] B. Lian and A. Linshaw, *Howe pairs in the theory of vertex algebras*, J. Algebra 317, 111-152 (2007).

- [LI] A. Linshaw, *Invariant theory and the $\mathcal{W}_{1+\infty}$ algebra with negative integral central charge*, J. Eur. Math. Soc. 13, no. 6 (2011) 1737-1768.
- [LII] A. Linshaw, *A Hilbert theorem for vertex algebras*, Transformation Groups, Vol. 15, No. 2 (2010), 427-448.
- [LIII] A. Linshaw, *Invariant theory and the Heisenberg vertex algebra*, Int. Math. Res. Notices, doi:10.1093/imrn/rnr171.
- [LZI] B. Lian and G. Zuckerman, *Commutative quantum operator algebras*, J. Pure Appl. Algebra 100 (1995) no. 1-3, 117-139.
- [P] C. Procesi, *The invariant theory of $n \times n$ matrices*, Advances in Math. 19 (1976), no. 3, 306–381.
- [We] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, 1946.

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